Int. J. Math. & Stat. Sci.
Vol. 3, No. 2, December 1994, 205–216

URN SCHEME TO OBTAIN PROPERTIES OF STIRLING NUMBERS OF SECOND KIND

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Abstract

Using the capture/recapture urn model described in Leit & Pereira (1987) several interesting properties of Stirling numbers of second kind are easily obtained. We follow the same lines of Yamato (1990) that described an urn model for the Stirling numbers of first kind

1991 A.M.S. Subject Classification: Primary 60C05, Secondary 11Y99.

1. Introduction

The present article is the natural sequel of Yamato (1990) that describes an urn scheme to obtain properties of the Stirling numbers of first kind. Here we concentrate on the Stirling numbers of second kind that is related with the capture/recapture urn model described by Leite & Pereira (1987). The connection of Stirling numbers with statistical models are described and studied by Charalambides & Singh (1988). The use of probabilistic urn models to obtain algebraic numbers is not new. For instance, an elementary case is the combinatorial number presented in Chapter 10 of Blackwell (1969).

Stirling numbers of second kind are the numbers $S(n;k)$ such that, for $n = 0, 1, \ldots,$

$$t^n = \sum_{k=0}^{n} S(n;k).(t)_k, \tag{1.1}$$

where $(t)_k = t(t-1)\ldots(t-k+1)$ for $k > 0$ and $(t)_0 = 1$. It is not difficult to prove that, using the definition, $S(0;0) = S(1;1) = 1$ and, for $n > 0, S(n;0) = 0$. By convention, we take $S(n;k) = 0$ for all $k > n$.

Section 2 describes the urn scheme considered, its probability distribution and its connection with the numbers $S(n;k)$. Using probability properties of this urn model, various results of Stirling numbers are obtained in Section 3. Section 4 introduces an interesting expression for the Stirling numbers. Most of the properties are well known and may be found in the literature. See for instance Comtet (1974) and Knuth (1973). All the quantities used in this paper are non-negative integers.

2. The Urn Scheme

Consider an urn with N white balls. A ball is randomly selected from the urn and it is replaced by a black ball. A second random selection from this new urn (1 black and $N-1$ white balls) is performed and if the ball selected is black it returns to the urn but if it is white, a black one would replace it.

This procedure is repeated sequentially until the n-th selection ($n \leq N$). That is, n independent selections are performed and in any of the selections whenever a white ball is selected a black ball would replace it and whenever a black ball is selected it is returned back to the urn.

For the i-th selection ($i = 1, 2, \ldots, n$) we define the random quantity U_i that takes value 0 if the ball selected is black and 1 if it is white. Hence, the total number of black (white) balls in the urn after $k (k \leq n)$ selections can be represented by $T_k = U_1 + \ldots + U_k (N - T_k)$. The probability distribution for the process is given by Leite & Pereira 1987.

$$P(U_1 = 1) = 1, P(U_1 = 1, U_2 = u_2) = \frac{(N-1)_{u_2}}{N} \text{ and for } n > 2,$$

$$P(U_1 = 1, U_2 = u_2, \ldots, U_n = u_n) = \frac{(N)_t}{N^n} \prod_{k=2}^{n-1} \binom{t_k}{1 - u_{k+1}}, \tag{2.1}$$

where $t_k = 1 + u_2 + \ldots + u_k$ and $t = t_n$.

The following result shows the relationship between this probability model and Stirling numbers of the second kind.

Lemma 2.1

The probability function of the random variable T_n is

$$P(T_n = t) = \frac{(N)_t}{N^n} S(n; t), \tag{2.2}$$

for $n = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, n$.

Proof. Note that $P(T_n = t) = \sum P(U_1 = 1, U_2 = u_2, \ldots, U_n = u_n)$, where the sum is over the set $\Omega = \{(u_1, u_2, \ldots, u_n); u_1 = 1; u_k = 0 \text{ or } 1 \text{ and } t = u_1 + \ldots + u_n\}$. Using now (2.1) we can write

$$P(T_1 = 1) = P(U_1 = 1) = 1, P(T_2 = t) = \frac{(U-1)_{t_1}}{N} \text{ and for } n > 2,$$

$$(2.3) \quad P(T_n = t) = \frac{(N)_t}{N^n} \sum \prod_{k=2}^{n-1} \binom{t_k}{1 - u_{k+1}} = \frac{(N)_t}{N^n} p(n,t)$$

or equivalently $N^n P(T_n = t) = (N)_t p(n,t)$. Taking the sum over the possible values of t in both sides of this expression we have

$$N^n \sum_{t=1}^{n} P(T_n = t) = N^n = \sum_{t=1}^{n} p(n,t)(N)_t.$$

This last expression shows that by definition $p(n,t)$ is the Stirling number of the second kind, $S(n;t)$. Using it in (2.3) we obtain (2.2).

Corollary 2.2

i) $S(n;1) = 1$ for $n \geq 1$;

ii) $S(n;n) = 1$ for $n \geq 0$;

iii) $S(n;2) = 2^{n-1} - 1$ for $n \geq 2$;

iv) $S(n;n-1) = \binom{n}{2}$ for $n \geq 2$.

Proof. For all items we consider that $n \leq N$. However since N is an arbitrary integer, this is not a restriction.

i) $P(T_n = 1) = P(U_1 = 1, U_2 = 0, \ldots, U_n = 0) = (1/N)^{n-1}$. Hence, $S(n;1) = P(T_n = 1)\dfrac{N^n}{(N)_1} = 1$.

ii) $S(0;0) = 1$ by definition. From (2.2) we can write

$$\begin{aligned} P(T_n = n) &= P(U_1 = 1, U_2 = 1, \ldots, U_n = 1) \\ &= \frac{(N)_n}{N^n} = \frac{(N)_n}{N^n} S(n;n). \end{aligned}$$

iii) Using now (2.2) and (2.3) we have

$$P(T_n = 2) = \frac{(N)_2}{N^n}\left(2^{n-2} + 2^{n-3} + \ldots + 2 + 1\right)$$
$$= \frac{(N)_2}{N^n}\left(2^{n-1} - 1\right) = \frac{(N)_2}{N^n} S(n;2).$$

iv) As above we have

$$P(T_n = n-1) = \frac{(N)_{n-1}}{N^n}(1 + 2 + 3 + \ldots + (n-1))$$
$$= \frac{(N)_{n-1}}{N^n}\frac{n(n-1)}{2} =$$
$$= \frac{(N)_{n-1}}{N^n}\binom{n}{2} = \frac{(N)_{n-1}}{N^n} S(n; n-1).$$

We end this section with a simple and useful result that we have not seen elsewhere.

Corollary 2.3

For $n \geq 3$ and $t \leq n$ we have

$$S(n;t) = \frac{1}{(t-1)!}\sum (t_2 t_3 \ldots t_{n-1}), \tag{2.4}$$

where, $t_k = 1 + u_2 + \ldots + u_k (k = 2, 3, \ldots, n-1)$ and the sum is over the set Ω,

$\Omega = \{(u_1, u_2, \ldots, u_n) | u_1 = 1; u_k = 0 \text{ or } 1 \text{ and } t = u_1 + \ldots + u_n\}.$

Proof. The proof is straightforward since it is not difficult to see that

$$P(T_n = t) = \frac{(N)_t}{N^n}\frac{1}{(t-1)!}\sum (t_2 t_3 \ldots t_{n-1})$$

3. Properties

In this section standard properties of the Stirling numbers of the second kind are obtained. In all the properties presented we are considering as initial conditions $S(0;0) = 1, S(n;0) = 0$ for $n > 0$, and $S(0;t) = 0$ for $t > 0$.

Proposition 3.1.
For $n \geq 0$ and $1 \leq t \leq n+1$ we have

$$S(n+1;t) = S(n;t-1) + tS(n;t) \tag{3.1}$$

Proof. From probability properties

$$\begin{aligned} P(T_{n+1} = t) &\\ &= P(T_n = t, U_{n+1} = 0) + P(T_n = t-1, U_{n+1} = 1) \\ &= P(U_{n+1} = 0 | T_n = t) P(T_n = t) \\ &+ P(U_{n+1} = 1 | T_n = t-1) P(T_n = t-1) \\ &= \left(\frac{t}{N}\right) P(T_n = t) + \left(1 - \frac{t-1}{N}\right) P(T_n = t-1). \end{aligned} \tag{3.2}$$

Using Lemma 2.1

$$\begin{aligned} S(n+1;t) &= \frac{N^{n+1}}{(N)_t} \left(\left(\frac{t}{N}\right) P(T_n = t) \right. \\ &\left. + \left(1 - \frac{t-1}{N}\right) P(T_n = t-1) \right) \end{aligned}$$

Using again Lemma 2.1, to write P as function of S, and after simple algebraic simplifications we obtain the result.

The following result is a probabilistic result that is proved simply by writing the event $\{T_{n+1} = t+1\}$ as the following union of disjoint events:

$$\begin{aligned} \{T_{n+1} = &\ t+1, T_n = t\} \cup \{T_{n+1} = T_n = t+1, T_{n-1} = t\} \cup \\ &\cup \{T_{n+1} = T_n = T_{n-1} = t+1, T_{n-2} = t\} \cup \ldots \\ &\ldots \cup \{T_{n+1} = T_n = T_{n-1} = \ldots = T_{t+1} = t+1, T_t = t\} \end{aligned}$$

Lemma 3.2

The probability function of the random variable T_n is recursively written as

$$P(T_{n+1} = t+1) = \left(\frac{N-t}{N}\right) \sum_{k=t}^{n} \left(\frac{t+1}{N}\right)^{n-k} P(T_k = t), \tag{3.3}$$

where $1 \leq t \leq n \leq N$.

Using this probabilistic result we obtain the following interesting property of the Stirling numbers:

Proposition 3.3

For $n \geq 0$ and $0 \leq t \leq n$ we have

$$S(n+1; t+1) = \sum_{k=t}^{n} (t+1)^{n-k} S(k; t) \tag{3.4}$$

Proof. Using Lemma 2.1 we write $S(n+1; t+1)$ as a function of $P(T_{n+1} = t+1)$ and then using Lemma 3.2 we conclude the result.

The following result is standard and presents $S(n; t)$ as a function of the $S(n+1; k)'s$ for $k > t$.

Proposition 3.4.

For $n \geq 0$ and $0 \leq t \leq n$ we have

$$S(n; t) = \sum_{k=t}^{n} (-1)^{k-t} (k)_{k-t} S(n+1; k+1) \tag{3.5}$$

Proof. Using expression (3.2) we obtain

$$\begin{aligned} P(T_{n+1} = k+1) = {} & \left(\frac{k+1}{N}\right) P(T_n = k+1) \\ & + \left(1 - \frac{k}{N}\right) P(T_n = k), \end{aligned} \tag{3.6}$$

for $t \leq k \leq n$. From (2.2) we write $S(n+1;k+1)$ as a function of $P(T_{n+1} = k+1)$ and them using (3.6) in the right side of (3.5) we conclude the result.

To obtain the final results of this section, we consider the random variable V that is the number of times that the first ball selected appears in the n trials. The sample space of V is then $\{1, 2, \ldots, n\}$ and

$$P(T_n = t) = \sum_{k=1}^{n-t+1} P(T_n = t, V = k).$$

Observe that the occurrence of the event $\{T_n = t, V = k\}$ is equivalent to "the first ball is selected exactly k times and $t-1$ white balls are selected in the remaining $n-k$ trials." Since there are $N\binom{n-1}{k-1}$ ways of selecting the first ball k times and, from (2.2), there are $(N-1)_{t-1}S(n-k;t-1)$ ways of selecting $t-1$ white balls in the remaining $n-k$ trials, we have that

$$\begin{aligned} P(T_n = t, V = k) &= \frac{N\binom{n-1}{k-1} S(n-k;t-1)(N-1)_{t-1}}{N^n} \\ &= \frac{\binom{n-1}{k-1} S(n-k;t-1)(N)_t}{N^n} \end{aligned}$$

and consequently

$$\begin{aligned} (3.7)\quad P(T_n = t) &= \sum_{k=1}^{n-t-1} P(T_n = t, V = k) \\ &= \sum_{k=1}^{n-t-1} \frac{\binom{n-1}{k-1} S(n-k;t-1)(N)_t}{N^n} \end{aligned}$$

$$= \sum_{k=0}^{n-t} \frac{\binom{n-1}{k-1} S(n-k-1;t-1)(N)_t}{N^n}.$$

Proposition 3.5.

For $1 \leq t \leq n$ we have

$$S(n;t) = \sum_{k=0}^{n-t} \binom{n-1}{k} S(n-k-1;t-1). \tag{3.8}$$

Proof. Using (2.2) in (3.7) the result follows.

Corollary 3.6

For $n \geq 0$ and $0 \leq t \leq n$ we have

$$S(n+1;t+1) = \sum_{k=t}^{n} \binom{n}{k} S(k;t). \tag{3.9}$$

Proof. The proof is straightforward if we use (3.8) properly.

4. An Useful Expression

This section presents an alternative expression for the Stirling number. It may be used to simplify some calculations. Before we present the result we introduce an important probabilistic result. Denote by $P_t(n, N)$ the probability that exactly t different balls are not observed in the n trials.

Lemma 4.1

For $N - n \leq t \leq N - 1$ we have

$$P_t(n,N) = \frac{\binom{n}{t}}{N^n} \sum_{k=0}^{N-t} (-1)^k \binom{N-t}{k} (N-t-k)^n. \tag{4.1}$$

Proof. Consider the event "exactly t (fixed) balls are not observed during the selection process $(N - n \leq t \leq N - 1)$". The occurrence of this event implies that $N - t$ distinct balls are selected in the n trials. Hence, the number of possible cases for this event is $(N-t)^n P_0(n, N-t)$ since $(N-t)^n$ is the number of ways we may select (with replacement) n balls among $N-t$. Recall that $N - t \leq n$. Also,

$$\binom{N}{t}$$

is the number of ways we can choose t balls to be fixed. Hence we can write

$$P_t(n,N) = \frac{\binom{N}{t}(N-t)^n P_0(n, N-t)}{N^n}. \tag{4.2}$$

To conclude the result we need to calculate $P_0(n, N-t)$. Consider the selection of n balls from an urn containing $N-t$ balls. Consider, for $k = 1, 2, \ldots, N-t$, the event A_k that corresponds to "the k-th ball is not selected at all in the n trials". The probability that at least one of the $N-t$ balls is not selected in the process is given by

$$\begin{aligned} P\left(\bigcup_{k=1}^{N-t} A_k\right) &= \sum_{k=1}^{N-t} P(T_n = t, V = k) \\ &= \sum_{k=1}^{N-t}(-1)^{k-1}\sum_i P(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_i}), \end{aligned}$$

where $\sum_i$ is the sum over the set $\{(k_1, k_2, \ldots, k_i) : 1 \leq k_1 \leq k_2 \leq k_i \leq N-t\}$. Since

$$\Sigma_i P(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_i}) = \Sigma_i \frac{(N-t-i)^n}{(N-t)^n},$$

with some simplification we obtain

$$P\left(\bigcup_{k=1}^{N-t} A_k\right) = \frac{1}{(N-t)^n}\sum_{k=1}^{N-t}(-1)^{k-1}\binom{N-t}{k}(N-t-k)^n.$$

Noting that $1 - P\left(\bigcup_{k=1}^{N-t} A_k\right) = P_0(n, N-t)$ and applying this result in (4.2) we obtain the result.

Finally, an analytical expression for $S(n;t)$ can be presented. For computational purposes it may be very useful.

Proposition 4.2.

For $n \geq 0$ and $0 \leq t \leq n$ we have

$$S(n;t) = \frac{1}{t!}\sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k}k^n. \tag{4.3}$$

Proof. It is not difficult to rewrite (4.1) as

$$\begin{aligned} P(T_n = t) &= P_{N-t}(n, N) \\ &= \frac{(N)_t}{t!N^n}\sum_{k=0}^{t-k}(-1)^{t-k}\binom{t}{k}k^n \end{aligned} \tag{4.4}$$

Applying (2.2) in (4.4) we obtain the final result.

We conclude this article by saying that the results presented here for Stirlings numbers are not new. However the use of the urn scheme simplifies the obtaintion of these results.

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Received by P.N.Rathie *on* August 31, 1993. Revised May 5, 1995.

Refereed by Clarice Freire, IMECC, UNICAMP, 13081, Campinas, S.P., Brasil .

Accepted on July 17, 1995