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Can Randomization be Informative?

Carlos A. B. Pereira, Thiago F. Campos, Gustavo M. Silva and Sergio Wechsler

Instituto de Matemática e Estatística - Rua do Matão, 1010 - Cidade Universitária - São Paulo - SP - Brasil - CEP 05508-090

Abstract. In this paper the Pair of Siblings Paradox introduced by Pereira [1] is extended by considering more than two children and more than one child observed for gender. We follow the same lines of Wechsler et al. [2] that generalizes the three prisoners' dilemma, introduced by Gardner [3].

This paper's conjecture is that the Pair of Siblings and the Three Prisoners dilemma are dual paradoxes. Looking at possible likelihoods, the sure (randomized) selection for the former is non informative (informative), the opposite that holds for the latter. This situation is maintained for generalizations. Non informative likelihood here means that prior and posterior are equal.

Keywords: Randomization, Three Prisoners Paradox, Monty Hall, Information
PACS: 02.50.-r

INTRODUCTION

The literature is full of simple situations for which the conditional probabilities bring paradoxical situations; for instance the classical Simpson's or the Borel's paradoxes. Among these situations we focus here in the comparison between The Three Prisoners and The Pair of Siblings Paradoxes. The former was introduced by Gardner [3] and the latter by Pereira [1]. We show that there is a duality when comparing these two situations.

The following section presents the Three Prisoners paradox (equivalent to the Monty Hall's paradox.) The third section reviews the Pair of Siblings paradox as introduced by Pereira [1]. It is argued about its duality with the Three Prisoner's dilemma. The fourth section extends the pair of siblings problem for more than two children and more than one child observed for gender. The fifty section presents our final remarks about information and the possible duality.

THE PRISONERS DILEMMA

The Monte Hall Paradox as presented by Morgan et al. [5] is as follows:

> *Suppose you are on a game show and given a choice of three doors. Behind one of them is a car; behind the remaining ones are goats. You pick door number 1 and the host, who knows what is behind the doors, opens door number 3, which has a goat. The host then asks if you want to switch your choice to door number 2. Should you switch?*

XI Brazilian Meeting on Bayesian Statistics
AIP Conf. Proc. 1490, 252-258 (2012); doi: 10.1063/1.4759609

You may keep your original choice since you see only two doors and may think that your chances are equal for both doors. However, if you believe that the host choice is at random when you would be in the right choice, then you must switch since your original probability, $1/3$, did not change: that is the probability the car is in the door number 2 becomes $2/3$. Morgan et al. [5] gives many answers to that question but some of them are under more assumptions than the ones given and they seem to be not quite correct. In fact both answers switch and not switch may be right depending of the assumptions to define the likelihood.

An equivalent and older paradox is the three prisoners' problem that is described and discussed in the sequel. This problem appeared in Martin Gardner's Mathematical Games column in Scientific American [3] [4] as follows:

Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin if you wish to decide whether to name B or C."
The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from $1/3$ *to* $1/2$*, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of* $1/3$ *to be the pardoned one, but his chance has gone up to* $2/3$*. What is the correct answer?*

These two paradoxes have received a lot of attention in the literature as in Dawid and Dickey [6] and Morgan et al. [5]. A painstaking extension of the prisoners' problem was introduced by Wechsler et al. [2] that consider N prisoners, k executions and m announcements.

Taking now p as the probability of the warden says "B is to be executed" if A is to be pardoned - the likelihood evaluated at the state of nature "A is to be pardoned" - and using the Bayes operator as in Wechsler et al. [2], the posterior probability of "A to be pardoned" given "B to be executed" is

$$P = \frac{p}{1+p}. \tag{1}$$

Figure 1 illustrates the behavior of P for all values of p. Note that when $p = 1$ the posterior is $P = 1/2$, case of prisoner A be correct and if $p = 1/2$ the posterior is $P = 1/3$, case of prisoner C be correct.

Considering now that the cells are named A, B and C as their corresponding inmates, if A and C could switch cells, we have the equivalence with the Monty Hall's problem.

A PAIR OF SIBLINGS

The Pair of Siblings paradox as presented by Pereira [1] is as follows:

The neighborhood dance school participates yearly in a national competition *with its group of 10 girls. Lily, the teacher, has learnt that the family of the two sisters that belong to the group will emigrate soon to another country. She heard that a new family, with two siblings, is moving to the apartment of the two ballerinas. Lily thinks that she could have another two dancers if both new kids are females. She could train them to join the group.*

Lily assigned a probability of $1/4$ *to the event of two new girls in the neighborhood. For her, the sample space was* $\{(m;m),(m;f),$ $(f;m),(f;f)\}$*, m and f representing male and female. Clearly, for her, every sample point had the same probability* $1/4$*. When talking to Jony, her brother who handles the rent of the apartment to the new family, she learned that at least one of the kids is female. In fact he was at the telephone talking to the kids' mother when she shouted to someone saying "be quiet girl" saying that she was talking to one of her kids. Lily then was happy since her space now become a set of three equally probable sample points,* $\{(m;f),(f;m),(f;f)\}$*; i.e., she now considers that her probability of* $(f;f)$ *is* $1/3$*.*

As soon Lily reaches this conclusion, Jony's wife, Mary, enters the office and said that a daughter of the new family was downstairs in her car. Lily then runs to the car and was happier since her probability now became $1/2$*. This conclusion arose from the fact that she looked for the gender of the other kid and the sample space became* $\{m,f\}$*, a set of two equally probable points. To accept Lily's analysis one should agree that the event "at least one sibling is a female" produces different probabilities depending on whether it is learned by different channels: auditory or visual.*

Is Lily correct in changing her probabilities whenever changing the channel of receiving information?

The statistician's eyes allow the incorporation of other kinds of information in the learning process. Let us consider all the nuances of the process. The parameter of interest takes the values $\theta = 1$ if the state of nature is $(f;f)$ and $\theta = 0$ otherwise. Considering the three point sample space, let the prior probability be $p(\theta = 1) = 1/3$. For the sample observation, let $x = 0$ if Mary had brought a male kid and $x = 1$ a female. The likelihood function is $L(\theta|x) = p(x|\theta)$, the sample probability function as a function of θ for any fixed observed x. Suppose that it takes the values

$$L(1|1) = 1 \text{ and } L(0|1) = q \in [0;1]. \tag{2}$$

The reader should confirm that the Lily's posterior probability of interest is

$$p(\theta = 1|x = 1) = (1+2q)^{-1}. \tag{3}$$

Lily was correct when considering that $q = 1/2$ producing $1/2$ as the posterior probability. However, after listening to the Lily's arguments, Mary reported that the mother had asked her to bring the girl to try out a skirt in the female clothing store next door. In that case she should consider that $q = 1$ and then the posterior would be $1/3$ just like the prior probability of $(f;f)$.

Figure 2 (Figure 1) shows a decreasing (increasing) function of the likelihood value q (p). The Random choice in both cases corresponds to $p = q = 1/2$. However, although

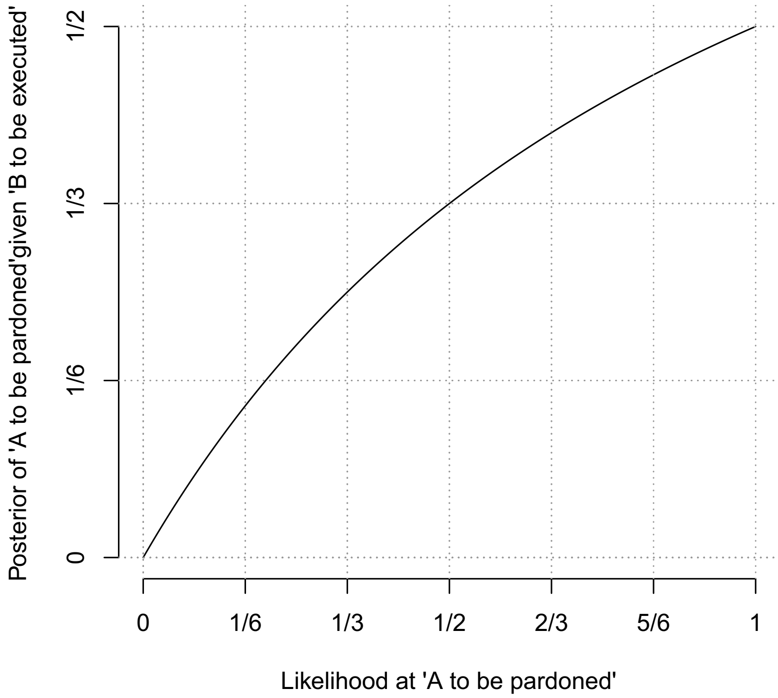

FIGURE 1. Posterior probability that prisoner A is to be pardoned given that the warden tells that B is to be executed

in the prisoners case the posterior equal to prior, $1/3$, here we move from $1/3$ to $1/2$. We recover the prior in fact when we consider the sure female choice ($q = 1$) to go in the car. At the sure choice in the prisoners case, $p = 1$, the posterior is going to be $1/2$. These facts in mind lead us to think that there is a duality among the two paradoxes. We can answer now the question posed by the title of this article: the randomized choice can make the prior different of the posterior making it informative. In addition, the sure choice can be uninformative as show in the siblings' case.

MANY SIBLINGS

Wechsler et al. [2] extended the Three Prisoners Paradox by considering N prisoners, k executions and m announcements. To extend the Siblings Problem we consider the case of N siblings, among them θ females and a sample (children in the car) of n kids with x females.

It is not difficult to understand that to randomly choose the children that go in the car we will have as the statistical model the Hypergeometric distribution with N as the

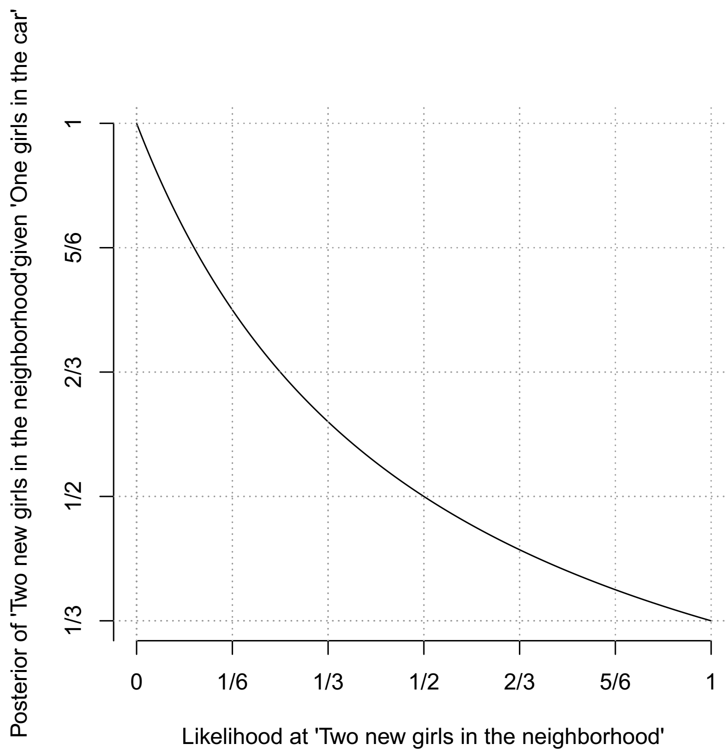

FIGURE 2. Posterior probability that we have two female siblings, i. e. $(f;f)$, given the female gender of the kid at the car, varying q

population, n as the sample, θ as the parameter and x as the observation: That is,

$$P(x|\theta) = \frac{\binom{\theta}{x}\binom{N-\theta}{n-x}}{\binom{N}{n}}.$$

Clearly this holds if $n - N + \theta \leq x \leq \theta$, and it is equal to zero otherwise.

We argue here that the natural prior for siblings' gender is the binomial with parameters N and $1/2$. That is,

$$P(\theta) = \binom{N}{\theta}\left(\frac{1}{2}\right)^N.$$

Consequently, the posterior probability function of θ is proportional to the following simple function:

$$P(\theta|x) \propto \binom{n}{x}\binom{N-n}{\theta-x}.$$

Clearly, as above, we have to observe all the limits of variability of the sample and parameter.

The likelihood version of the sure choice in this extension is going to be

$$L(\theta|x) = \begin{cases} P(x=i|\theta=i)=1 \text{ and } P(x\neq i|\theta=i)=0, & \text{if} \quad i\leq n; \\ P(x=n|\theta=i)=1 \text{ and } P(x\neq n|\theta=i)=0, & \text{if} \quad i>n. \end{cases} \tag{4}$$

Note that, if for some reason we learn that $\{\theta \geq n\}$, then simply the posterior is equal to prior for all alternatives and the sure choice is non informative, contrarily to the three prisoners case.

To illustrate this extension we consider the case of $N=6$ and $n=3$ which results are presented in Table 1 and 2.

TABLE 1. Likelihood of the random choices (sure choice)

θ values	x values			
	0	1	2	3
0	1 (1)	0 (0)	0 (0)	0 (0)
1	0.5 (0)	0.5 (1)	0 (0)	0 (0)
2	0.2 (0)	0.6 (0)	0.2 (1)	0 (0)
3	0.05 (0)	0.45 (0)	0.45 (0)	0.5 (1)
4	0 (0)	0.2 (0)	0.6 (0)	0.2 (1)
5	0 (0)	0 (0)	0.5 (0)	0.5 (1)
6	0 (0)	0 (0)	0 (0)	1 (1)

TABLE 2. Posterior probabilities (proportional) under random choice (sure choice)

θ values	x values			
	0	1	2	3
0	1 (1)	0	0	0
1	3 (0)	3 (1)	0	0
2	3 (0)	9 (0)	3 (1)	0
3	1 (0)	9 (0)	9 (0)	1 (0.48)
4	0	3 (0)	9 (0)	3 (0.36)
5	0	0	3 (0)	3 (0.14)
6	0	0	0	1 (0.02)

Looking at Table 2, last column inside the brackets, we identify that we have the prior probability function conditional to the event $\{\theta \geq 3\}$. Agreeing with the fact that the sure choice is uninformative if one happens to know that $\theta \geq 3$.

FINAL REMARKS

The objective of this paper is to show that there are cases where the random choice is more informative than a sure choice. With the siblings case although the random choice is more informative than the sure choice, it is not the best likelihood - the one that provides more change from prior to posterior -. We have a better choice whenever the likelihood value $L(\theta=0|x=1)=q<1/2$. Note that the random choice, $q=1/2$, is better for any choice of $q \in (1/2;1]$. There are non randomized choices that are better or worse than the randomized one: it is in the middle way as can be seen in Figure 2.

REFERENCES

1. C. A. B. Pereira, "Standard statistical concepts: can they produce incoherence?," in *ICOTS-7 Conference Proceedings on CD*, edited by I. A. for Statistical Education, In: Allan Rossman & Beth Chance (Org.), Amsterdam, 2006, vol. 3I, pp. 18–24, 7 ed. edn.
2. S. Wechsler, L. G. Esteves, A. Simonis, and C. Peixoto, *Synthese* **143**, 255–272 (2005).
3. M. Gardner, *Scientific American* pp. 180–182 (1959).
4. M. Gardner, *Scientific American* p. 188 (1959).
5. J. P. Morgan, N. R. Changanty, R. C. Dahiya, and M. J. Doviak, *The American Statistician* **45**, 284–287 (1991).
6. A. P. Dawid, and J. M. Dickey, *Journal of the American Statistical Association* **72**, 845–850 (1977).