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Bayesian Estimation of Component Reliability in Coherent Systems

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ABSTRACT The first step in statistical reliability studies of coherent systems is the estimation of the reliability of each system component. For the cases of parallel and series systems, the literature is abundant, but it seems that the present paper is the first to present the general case of component inferences in coherent systems. The failure time model considered here is the three-parameter Weibull distribution. Furthermore, identically distributed failure times are not a required restriction. An important result is proved: without the assumption that components' lifetimes are mutually independent, a given set of sub-reliability functions does not identify the corresponding marginal reliability function. The proposed model is general in the sense that it can be used for components of any coherent system, from the simplest to the most complex designs. It can be considered for all kinds of censored data, including interval-censored data. An important property obtained for the Weibull model is that the posterior distributions are proper, even for non-informative priors. Using several simulations, the excellent performance of the model is illustrated. As real examples, boys' first use of marijuana and a device from a field-tracking data set are considered to show the efficiency of the solution even when censored data occurs.

INDEX TERMS Bayesian paradigm, bridge system, coherent system, component lifetime, parallel system, parametric estimation, series system, Weibull model, 2-out-of-3 system.

I. INTRODUCTION

As motivation, the reliability estimation of a device with two components in series is considered [1]. The device has two possible causes of failure: 1) an electrical surge failure or 2) a wear-out failure. This is a field-tracking study, and the cause of failure can be observed. Clearly, the failure of either component 1 or 2 leads to device failure. Here, the failure of a component implies that the possible future failure time of the other becomes invisible, i.e., censored data. The statistical inference for the reliability of the device depends on both marginal components' probability models, even in the presence of censoring. In addition, it is important to estimate the lifetime distribution of a particular component for future system design and maintenance planning [2]. Hence, inferences for both components are needed.

Statistical inference of component reliability is not an easy task: censoring, dependence, and unequal distributions are some of the troubles. Considering a sample of the device discussed above, the n sample units are observed up to death. For each system unit, one of the components produces its failure time, and the remaining component lifetime is censored. It is reasonable to say that the two types of components are not identically distributed, since it is likely that one of the components may suffer more censoring than the other. For a system with m components, the last component to fail is the responsible for the system failure at time t (the time when the system failed), implying that all the remaining components are also censored at time t, although the types of censoring could be different (some components can be failed at time t, and some could be working). It is clear that all components are important for system reliability and are in some way responsible for the failure of the system. We define the component responsible for the system failure as the component whose failure time is not censored, that is, it was the last component to fail and then the system failed.

The reliability of a system and its components also depends on the system design: the way components are interconnected. The device example described previously is a series

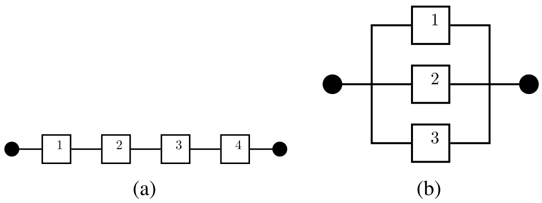


FIGURE 1. **(a) Representation of a series system with 4 components; (b) Representation of a parallel system with 3 components.**

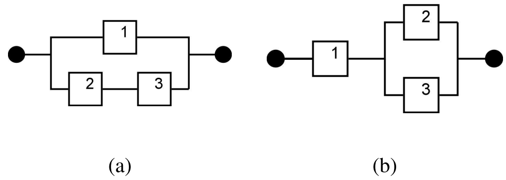


FIGURE 2. **(a) Parallel-series representation; (b) Series-parallel representation.**

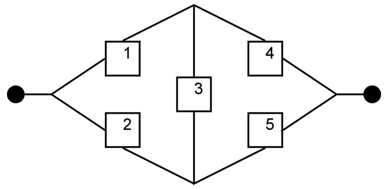


FIGURE 3. **Bridge design.**

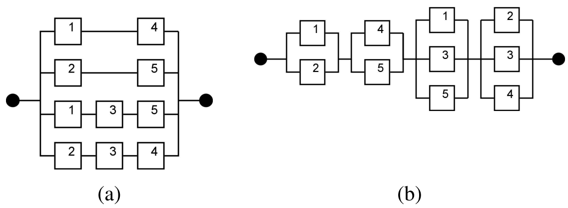


FIGURE 4. **(a) Bridge parallel-series representation; (b) Bridge series-parallel representation.**

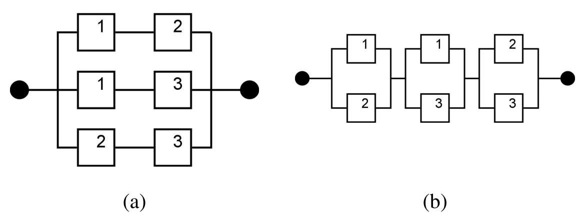


FIGURE 5. **(a) 2-out-of-3 parallel-series representation; (b) 2-out-of-3 series-parallel representation.**

system with two components, a simple case known as a competing risks problem. Figure 1a is a series system with four components. At the time the system fails, only one component is uncensored; the other three components are right-censored at the system failure time. A parallel system, as in Figure 1b, works whenever at least one component is working. Again, only one component has its failure time uncensored; the other components are left-censored observations.

The literature on the reliability of either parallel or series systems is abundant, and different solutions have been presented. For series systems, [3] treated the estimation problem, and [4] presented the relationship between the subdistribution functions of the components (the probability of the system working until a certain time and the interest component responsible for system failure). A nonparametric estimator for these functions and a modified Cox proportional hazard model for competing risks were developed by [5].

References [6]–[9] discussed the Bayesian nonparametric statistics for series and parallel systems. Under Weibull probability distributions, Bayesian inferences for system and component reliabilities were introduced by [10]. Reference [11] presented a hierarchical Bayesian Weibull model for component reliability estimation for both series and parallel systems, proposing a useful computational approach. Using simulation for series systems and considering Weibull families, [12] compared the following three types of estimates: Kaplan-Meier, maximum likelihood and Bayesian plug-in estimators. Reference [13] also performed a comparative study of survival function estimation. Considering the celebrated property that any coherent system can be written as a combination of parallel and series systems, [14] introduced Bayesian nonparametric statistics for a class of coherent systems. Figure 2 illustrates two cases of this kind of combination with three components. Component 2, for example, is susceptible to both right- and left-censoring. Reference [14] restricted themselves to cases for which no component appears more than once in parallel-series or series-parallel representations.

In general, for a coherent system that uses a representation combining parallel and series systems, some components may appear in two or more places. Figure 3 is the bridge system described in the literature [15], and Figure 4 illustrates its parallel-series and series-parallel combinations. Note that each of the five components appears twice for both representations. Another important design is the k-out-of-m system (it works only if at least k out of the m components work). Figure 5 considers the simple 2-out-of-3 case in parallel-series and series-parallel representations. Note that each of the three components also appears twice in both combinations.

In their nonparametric inferences for coherent systems, [16] restricted themselves to cases of independent and identically distributed failure times; hence, all components have the same reliability. The method introduced in the present article is not restricted to the assumption of identically distributed component lifetimes. Another restriction, not highlighted in most of the literature, is that the failures of any pair of components cannot occur at the same time. Here, this restriction is also unnecessary. Our important restrictions are that all component lifetimes are mutually independent and three-parameter Weibull distributed; this is a very general family of distributions that can approximate most lifetime parametric distributions. Besides the left- and right-censored observations, the interval-censored observations can also be handled here. An important result of this paper is that without the assumption that components' lifetimes are mutually independent, a given set of sub-reliability functions does not identify the corresponding marginal reliability function. An advantage of the Weibull is that in our approach, even with a class of improper priors, the posterior distributions are proper. The proposed mechanism of calculation can be well used for any other family of distributions whenever proper priors are used.

Section II describes the Bayesian Weibull model. Section III presents the simulation studies to show the excellent performance of the proposed model, and we compare the performance of our approach to that presented by [16]. Section IV illustrates the methodology by considering two practical motivations: a data set of systems with two components in series and a data set for which interval censoring appears. Final considerations appear in Section V. The Appendix shows the proofs of two important results. The first is that mutual independence of the components' lifetimes is necessary to avoid non-identifiability problems. The second is that the posterior distributions considered here are proper.

II. BAYESIAN WEIBULL MODEL

In a coherent system with m components, let X_j be the failure time of j-th component, $j = 1, \ldots, m$. The system failure time is denoted by $T = h(X_1, \ldots, X_m)$, in which $h(\cdot)$ is the function that relates the system failure time to the components' functioning, and it depends on the system design. For a series system, for example, $T = h(X_1, \ldots, X_m) = \min\{X_1, \ldots, X_m\}$. The indicator of the component whose failure caused the system to fail is $\delta = j$ when $T = X_j$, $j = 1, \ldots, m$.

The j-th sub-distribution function evaluated at the time t is the probability that the system survives at most to the time t and failure indicator is j-th component, that is, $Q_j(t) = \Pr\left\{(T \leq t) \cap (\delta = j)\right\}$.

Besides, let $G_j(t) = \Pr\left\{(T > t) \cap (\delta = j)\right\}$ be the j-th sub-reliability function evaluated at the time t. Denote by $F_j(t)$ the marginal distribution function (DF) for the j-th component evaluated at time t, and $R_j(t) = 1 - F_j(t)$ the marginal reliability function, $j = 1, \ldots, m$.

Our aim is to estimate the marginal distributions of components' lifetimes. Therefore, we assume that $X_1, \ldots, X_m$ are mutually independent. This assumption is necessary to avoid the non-identifiability problem of the model.

When the interest is only in the estimation of the marginal distributions of each component in series system, [17] shows that without the hypothesis that components' survival times are mutually independent, the model of survival times is unidentifiable: the set of sub-reliability functions is consistent with an infinity of joint reliability functions. The following theorem is the natural extension to coherent systems.

Theorem 1: Let $X_1, \ldots, X_m$ be the failure time of the m components involved in a coherent system. Without the assumption that $X_1, \ldots, X_m$ are mutually independent, a given set of sub-reliability functions (sub-distribution functions) does not identify the corresponding marginal reliability functions (distributions functions) when information from other components are not available or not necessary.

Proof: The proof of this result is given in the Appendix. ■

When a coherent system is written as series-parallel or parallel-series subsystems, it is possible that the subsystems share components with the original configuration. In this type of configuration (shared components), it is clear that the components are not independent, and the marginal distribution of all the components will not be identifiable. This is not only an independence problem, but also the set of jump points will not be disjoints. This is a limitation of the nonparametric estimator from [7]–[9]. Our method does not need the subsystem representation with shared component, and the proposed model has the assumption of continuous random variables, then the set of joint jump points has probability zero.

A simple random sample of n systems with the same design is observed, where $t_1, \ldots, t_n$ is a sample of the random variable T. The goal is to estimate the reliability of components. At system failure, however, not all components would have their failure time observed. In addition, a particular component may be responsible for system failures in some sample units and not in the remaining ones, which are cases of censoring on component failure time. The amount and the types of censoring depend on the design of the system. For example, a sample of n units of a machine with a configuration as in Figure 2a is observed: The failure of component 1 alone is not enough for system failure. On the other hand, every failure of the system implies the failure of component 1 either being left-censored or being the last to fail. The latter should imply that one of the other two components, say component 2, failed before component 1, a case of left-censoring, and component 3 is a right-censored observation.

When a system fails, the failure time of a given component j may not be observed, but its censored time of failure is. For all sample units, the system failure times $t_1, \ldots, t_n$ are recorded. For a specific component j that is not responsible for one of the n systems that failed at time t, either it is right-censored, in which case it could still continue to work after t, or it is censored to the left if it has failed before t. Another kind of censoring could also occur: Suppose a machine failure time (a sample unit) is in an interval (L, U), L for the observed lower limit and U for the upper limit. If two or more components failed, they are all interval-censored in (L, U). To generalize the notation for all cases of component failure and censoring, consider the following notation: for a specific component j of system unit i, let (L_{ji}, U_{ji}) be a general interval of time in which

- $L_{ji} = U_{ji} = t_i$, if the j-th component failure time causes the i-th system failure time:
- $L_{ji} = t_i$ and $U_{ji} = \infty$ if the j-th component is right-censored at t_i;

- $L_{ji} = 0$ and $U_{ji} = t_i$ if the j-th component is left-censored at t_i;
- $0 < L_{ji} < U_{ji} < \infty$ if the j-th component is interval-censored.

It is worth noting that in our approach, it is not necessary to know the design of the system. The available information of each unit is the system failure time and the status of its components in the moment of system failure.

To complete the theoretical environment, let X_j be the random variable representing the j-th component failure time with density function $f(x_j|\boldsymbol{\theta_j})$ and with reliability function $R(x_j|\boldsymbol{\theta_j})$. $\boldsymbol{\theta_j}$ is the parameter, which can be either a scalar or a vector.

Using the above notation, the likelihood function is as follows:

$$L(\boldsymbol{\theta_j} \mid l_j, u_j) = \prod_{i=1}^{n} \left[f(l_{ji}|\boldsymbol{\theta_j}) \right]^{I_{\{l_{ji}=u_{ji}\}}} \times \left[R(l_{ji}|\boldsymbol{\theta_j}) - R(u_{ji}|\boldsymbol{\theta_j}) \right]^{1-I_{\{l_{ji}=u_{ji}\}}}, \quad (1)$$

where $I_{\{TRUE\}} = 1$ or $I_{\{FALSE\}} = 0$, $l_j = (l_{j1}, \ldots, l_{jn})$ and $u_j = (u_{j1}, \ldots, u_{jn})$.

The likelihood function in (1) is generic and straightforward for any probability distribution. The distribution considered here is the three-parameter Weibull. The choice of this distribution is due to the variation of parameter values implying changes in both distribution shape and hazard rates. We can have increasing, decreasing and constant failure rates in this family of Weibull distributions [18].

The Weibull reliability function is as follows:

$$R(x_j \mid \boldsymbol{\theta_j}) = \exp\left[-\left(\frac{x_j - \mu_j}{\eta_j} \right)^{\beta_j} \right],$$

for $x_j > 0$, where $\boldsymbol{\theta_j} = (\beta_j, \eta_j, \mu_j)$ and $\beta_j > 0$ (shape), $\eta_j > 0$ (scale) and $0 < \mu_j < x_j$ (location).

The Weibull distribution with two parameters ($\mu_j = 0$) is the most celebrated case in the literature. However, the location parameter μ_j that represents the baseline lifetime has an important meaning in reliability and survival analysis. In reliability analysis, a component under test may not be new. In medicine, for instance, a patient may have the disease before the onset medical appointment. Not taking into account this initial time can lead to an underestimation of the other parameters. Clearly, for new component testing, μ_j may be 0.

The estimation in this work is performed under a Bayesian perspective of inference, and thus, the a priori distribution to $\boldsymbol{\theta_j} = (\beta_j, \eta_j, \mu_j)$ needs to be defined.

There are situations where information about the functioning of the component through expert opinion and/or past experience can be expressed in the a priori distributions. In this work, no prior information about the functioning of the components is available, and the considered non-informative prior distribution in this work is

$$\pi(\beta_j, \eta_j, \mu_j) = \frac{1}{\eta_j} \frac{1}{\beta_j}. \quad (2)$$

The choice of the priori distribution in (2) guaranttes that the posterior distribution is a density function, as can be seen at Theorem 2.

The posterior density of $\boldsymbol{\theta_j} = (\beta_j, \eta_j, \mu_j)$, combining (1) and (2), comes out to be

$$\begin{aligned} &\pi(\beta_j, \eta_j, \mu_j \mid l_j, u_j) \\ &\propto \frac{1}{\eta_j} \frac{1}{\beta_j} \times \prod_{i=1}^{n} \Bigg\{ \left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j - 1} \frac{\beta_j}{\eta_j} \\ &\quad \times \exp\left[-\left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \Bigg\}^{I_{\{l_{ji}=u_{ji}\}}} \\ &\quad \times \Bigg\{ \exp\left[-\left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \\ &\quad - \exp\left[-\left(\frac{u_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \Bigg\}^{1-I_{\{l_{ji}=u_{ji}\}}} \end{aligned} \quad (3)$$

Even though (2) is not a proper prior (its integral is not finite), the posterior density in (3) is still a proper, as stated by the following result.

Theorem 2: Let a class of non-informative priors given by

$$\pi(\beta_j, \eta_j, \mu_j) = \frac{1}{\eta_j \beta_j^b}, \quad b \geq 0.$$

Even though for $b \geq 0$, $n = 1$ and the existence of a failure, the posterior density in (3) is not proper, and for $n > 1$, the posterior in (3) is proper.

Proof: The proof of this result is given in the Appendix. ■

The importance of the above result is that one can perform Bayesian inferences even with little prior information.

Because the posterior density in Equation (3) does not have a closed form (it is not possible to calculate the constant of proportionality in an analytical way), statistical inferences about the parameters can rely on Markov-Chain Monte-Carlo (MCMC) simulations. Here, we consider an adaptive Metropolis-Hasting algorithm with a multivariate distribution [19].

Discarding burn-in (first generated values discarded to eliminate the effect of the assigned initial values for parameters) and jump samples (spacing among generated values to avoid correlation problems), a sample of size n_p from the joint posterior distribution of $\boldsymbol{\theta_j}$ is obtained. For the j-th component, the sample from the posterior can be expressed as $(\beta_{j1}, \beta_{j2}, \ldots, \beta_{jn_p})$, $(\eta_{j1}, \eta_{j2}, \ldots, \eta_{jn_p})$ and $(\mu_{j1}, \mu_{j2}, \ldots, \mu_{jn_p})$. Consequently, posterior quantities of reliability function $R(t \mid \boldsymbol{\theta_j})$ can be easily obtained [20]. For instance, the posterior mean of the reliability function is

$$E[R(t \mid \boldsymbol{\theta_j}) \mid l_j, u_j] = \frac{1}{n_p} \sum_{k=1}^{n_p} R(t \mid \boldsymbol{\theta_{jk}}), \quad t > 0. \quad (4)$$

III. MODEL EVALUATION WITH SIMULATED DATA

To evaluate the quality of the model described above, this section presents simulation studies that are divided into two parts. First, the interest is in evaluating the effect of censorship in the quality of the proposed estimator. For this purpose, the proposed uncensored model is compared to its censored version in different sample sizes. Second, our approach is compared to the one presented by [16] in scenarios with different distributions to generate the lifetime of the components, the percentages of censor data and the types of complex system design.

The observed information is the failure time of the n observed systems and, for each unit that failed, the status of each component at the system failure time.

To generate the data of each simulated example, with m being the number of components and n the sample size, the following steps are considered.

For each system unit i, for $i = 1, \ldots, n$:

1) Draw X_{ji} from a given distribution for $j = 1, \ldots, m$;
2) Let $T_i = h(X_{1i}, \ldots, X_{mi})$, where T_i is the system failure time and $h(\cdot)$ is the function that relates the system failure time to the components' functioning, and it depends on the system design;
3) For each j component, where t_i is the system failure time:
   - If $X_{ji} = t_i$, then $l_{ji} = t_i$ and $u_{ji} = t_i$ is recorded;
   - If $X_{ji} < t_i$, then $l_{ji} = 0$ and $u_{ji} = t_i$ is recorded;
   - If $X_{ji} > t_i$, then $l_{ji} = t_i$ and $u_{ji} = \infty$ is recorded;
4) The data set for the j-th component is $\{(l_{j1}, u_{j1}), (l_{j2}, u_{j2}), \ldots, (l_{jn}, u_{jn})\}$.

To obtain posterior quantities, we used an MCMC procedure to generate a sample from the posterior distribution of the parameters. We generated 20, 000 samples from the posterior distribution of each parameter. The first 10, 000 of these samples were discarded as burn-in samples. A jump of size 10 was chosen to avoid correlation between the samples. The final sample size of the parameters generated from the posterior distribution was 1, 000. The chains' convergence was monitored in all simulation scenarios for good convergence results to be obtained. The posterior mean is considered the point estimator for the reliability function obtained through our Weibull model. We will denote the posterior mean as W3PM (Weibull 3-Parameter Model).

A. CENSORSHIP EFFECT IN DIFFERENT SAMPLE SIZES

In this first part of the simulation studies, we consider three simulated types of data:

- **Example 1**: Parallel system with $m = 4$ components, in which lifetimes are generated by a normal multivariate distribution truncated at 0 with mean vector (205, 210, 215, 211) and covariance matrix

$$\begin{bmatrix} 200 & 10 & 15 & 20 \\ 10 & 210 & 23 & 25 \\ 15 & 25 & 215 & 10 \\ 20 & 25 & 10 & 213 \end{bmatrix}.$$

- **Example 2**: 2-out-of-3 system in which X_1 was generated from a Weibull distribution with mean 15 and variance 8, X_2 from a gamma distribution with mean 18 and variance 12 and X_3 from a log-normal distribution with mean 20 and variance 10.
- **Example 3**: Bridge design in which X_1 was generated from a Weibull distribution with mean 17 and variance 8, X_2 from a log-normal distribution with mean 16 and variance 22, X_3 from a log-normal distribution with mean 15 and variance 15, X_4 from a gamma distribution with mean 15 and variance 6 and X_5 from a gamma distribution with mean 20 and variance 12.

For each simulated example, 40 data sets with different samples sizes were generated by considering n from 25 to 1000 by 25. The goal is to compare the sample size effect to the proposed estimator. For this purpose, the proposed uncensored model is compared to its censored version in different sample sizes.

For Example 1, the percentages of the censored data are 87% for component 1, 77% for component 2, 62% for component 3 and 74% for component 4. In Example 2, component 1 is censored in 77.8% of systems, component 2 is censored in 54.4% of systems, and component 3 is censored in 67.8% of systems. In the Example 3, components 1, 2, 3, 4 and 5 are censored in 80.1%, 69.3%, 90%, 80.8% and 79.8% of systems, respectively. As one can note, all components have high percentages of censored data.

A data set is generated for each sample size in each example, and we compare the mean absolute error (MAE) from the estimator in the censored scenario to the uncensored model. $R(t)$ and $\widehat{R}(t)$ are the uncensored reliability function and the estimate in the censored scenario, respectively. Hence, the MAE is evaluated by $\frac{1}{l}\sum_{\ell=1}^{l} \mid \widehat{R}(g_\ell) - R(g_\ell) \mid$, where $\{g_1, \ldots, g_\ell, \ldots, g_l\}$ is a grid in the space of failure times.

The MAE values are presented in figures 6, 7, and 8. As expected, the MAE values decrease as the sample sizes increase. In general, the MAE value stabilizes when $n = 200$ for the three examples.

B. SIMULATION STUDIES WITH DIFFERENT SCENARIOS

In this section, six scenarios with different generators of component lifetimes are considered. Also, different percentages of censored data are considered. Two types of system designs are used: a bridge system (Figure 3) and a 2-out-of-3 system (Figure 5). For each scenario, five different sample sizes of system units are considered (n = 25, 50, 100, 300, 1000).

The W3PM estimates are compared to the nonparametric estimates of [16], which we will call BSNP (Bhattacharya-Samaniego Nonparametric Estimator). Their approach can be used for all components involved in the reliability of any system, even for more-complex designs. The only necessary types of information for the computation of the estimates are the system design and the observed system failure times. However, in their work, there is the strong restriction that all the component lifetimes are mutually independent and identically distributed. Consequently, all components have

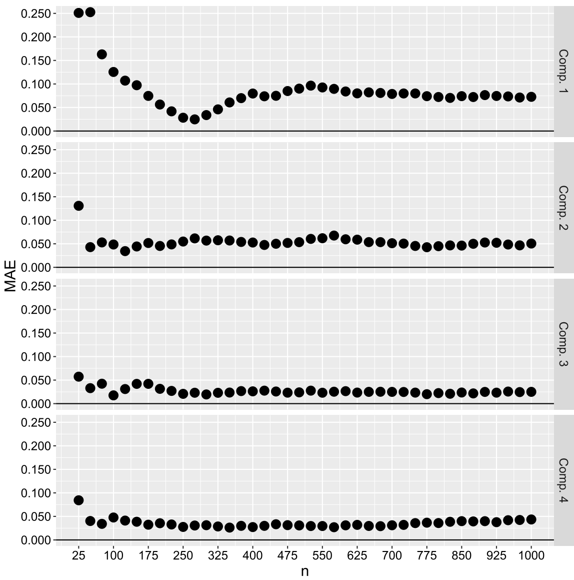


FIGURE 6. MAE values in scenarios with different sample sizes for components 1 to 4 in example 1.

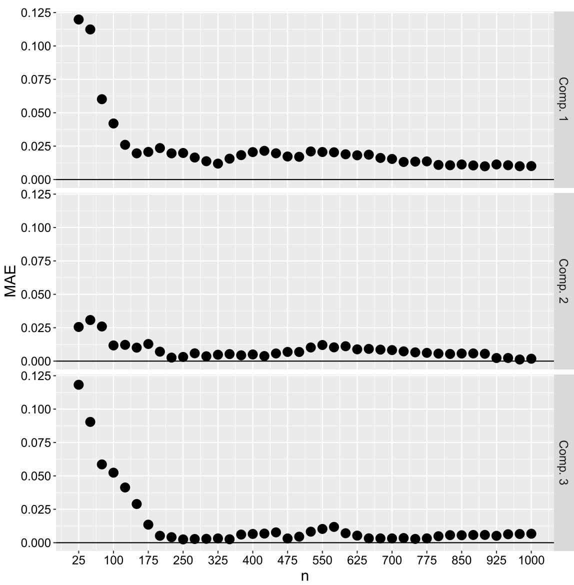


FIGURE 7. MAE values in scenarios with different sample sizes for components 1 to 3 in example 2.

the same reliability. The present approach does not have this limitation.

For each scenario, 1000 copies (data sets) are generated, and we evaluate the MAE from the estimators to the true distribution as the comparison measure.

Six scenarios are presented:

- **Scenario 1**: The same generation as example 2.
- **Scenario 2**: 2-out-of-3 design, in which X_1 was generated from a log-normal distribution with mean 4 and variance 7, X_2 was generated from a modified Weibull distribution [21] with mean 2.88 and variance 12.44,

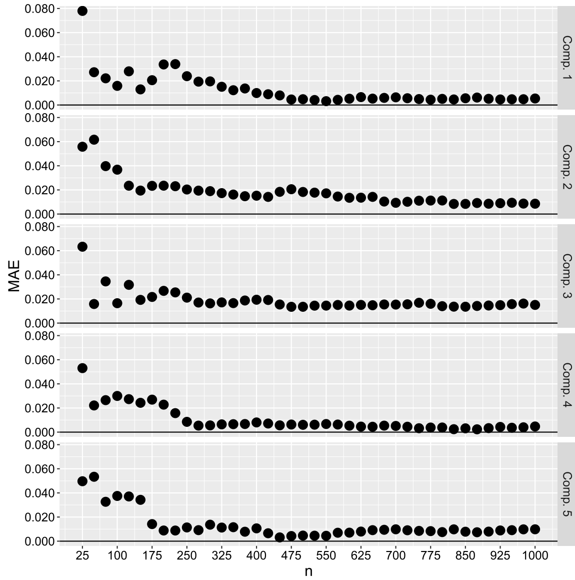
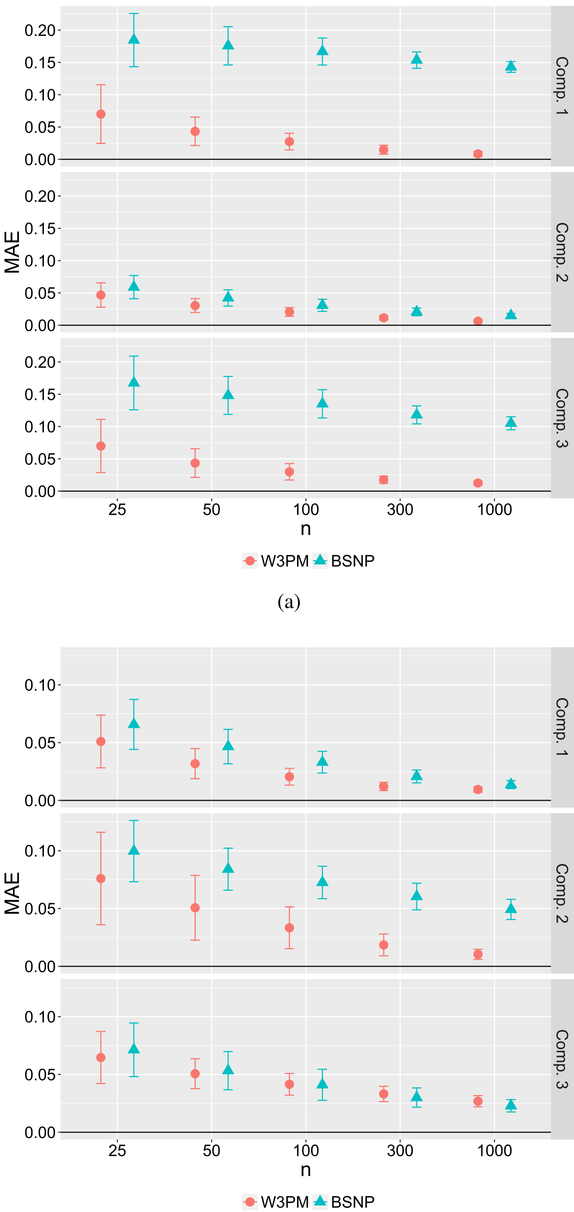


FIGURE 8. MAE values in scenarios with different sample sizes for components 1 to 5 in example 3.

and X_3 was generated from a three-parameter Weibull distribution with mean 5 and variance 3.
- **Scenario 3**: 2-out-of-3 design in which X_1, X_2 and X_3 were generated from Weibull distributions with means 10, 11, 10 and variances 2, 10, 8, respectively.
- **Scenario 4**: 2-out-of-3 design in which X_1, X_2 and X_3 were generated from modified Weibull distributions [21] with means 1.6, 2.4, 2.9 and variances 6, 4, 13, respectively.
- **Scenario 5**: The same generation as example 3.
- **Scenario 6**: Bridge design in which X_1 was generated from a Weibull distribution with mean 4 and variance 15, X_2 was generated from a modified Weibull distribution [21] with mean 5.6 and variance 15, X_3 was generated from a log-normal distribution with mean 6 and variance 7, X_4 was generated from a gamma distribution with mean 5 and variance 8, and X_5 was generated from a three-parameter Weibull distribution with mean 4 and variance 8.

Since a component that causes system failure causes the other components to become right- or left-censored data, the high percentages of censored data for all scenarios are shown in Table 1. It can be noted that all components in all scenarios have high percentages of censored data, with all cases being higher than 50%, reaching up to 90% (see component 3 in scenario 5).

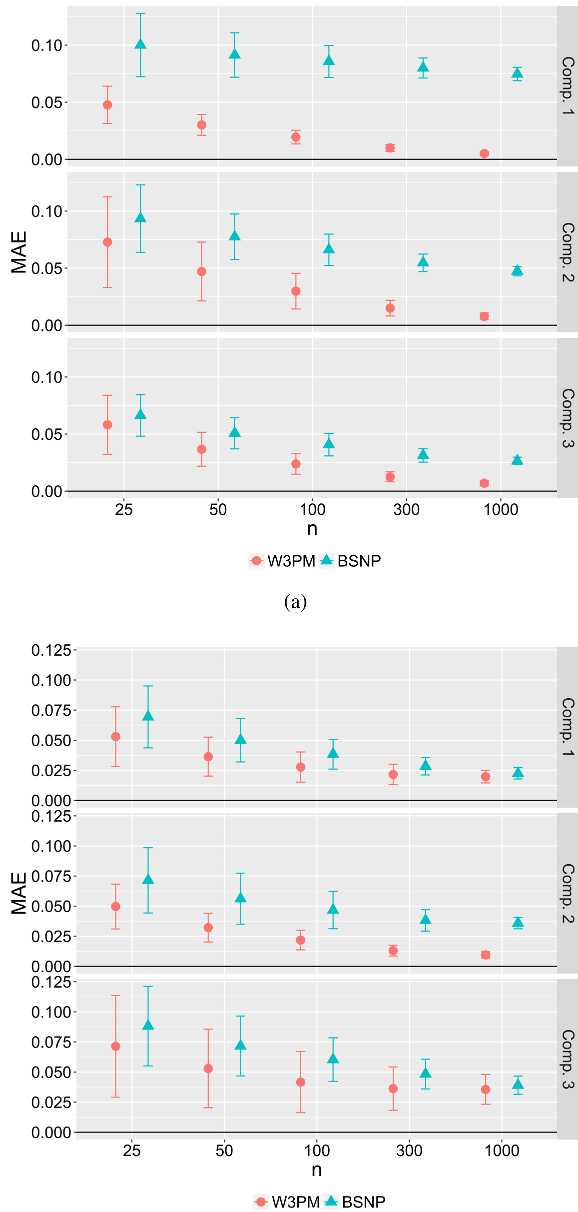
The mean and standard deviation of $1,000$ MAE values obtained for W3PM and BSNP are presented in figures 9a to 11b for scenarios 1 to 6. For component 3 from scenarios 2 and 4, the two estimation methods showed similar behavior. For the other situations, W3PM always presents a lower mean of MAE values, and the performance of the proposed estimator improves as n increases.



(a)

(b)

FIGURE 9. Mean (symbol) and standard deviation (bars) of the MAE of W3PM and BSNP for scenarios 1 and 2. W3PM indicates the posterior mean obtained by the proposed model, and BSNP indicates the nonparametric estimation proposed by [16]. (a) Scenario 1. (b) Scenario 2.



(a)

(b)

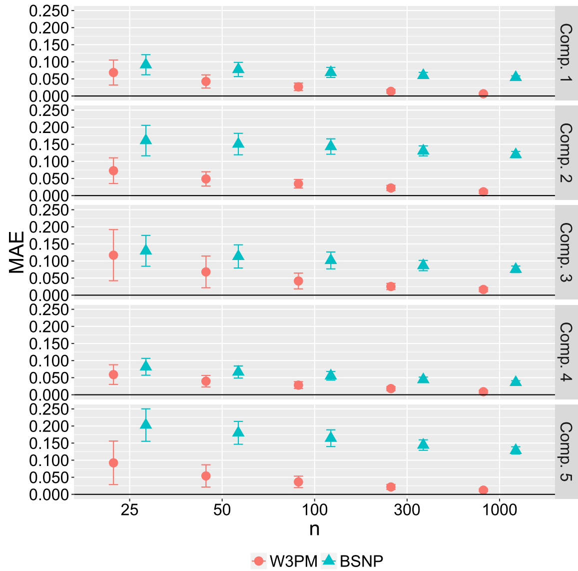
FIGURE 10. Mean (symbol) and standard deviation (bars) of the MAE of W3PM and BSNP for scenarios 3 and 4. W3PM indicates the posterior mean obtained by the proposed model, and BSNP indicates the nonparametric estimation proposed by [16]. (a) Scenario 3. (b) Scenario 4.

IV. APPLICATION

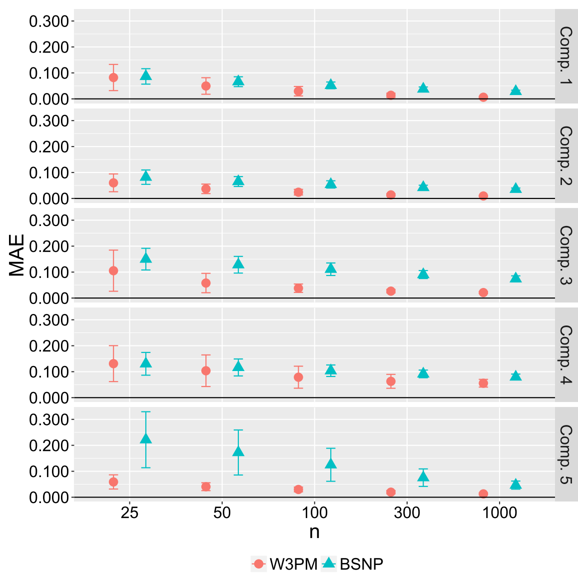
A. DEVICE FROM A FIELD-TRACKING DATA SET

To show the applicability of the proposed method, the reliability estimation of a device in typical service environments is presented [1]. The device presents two failure modes: an electrical surge failure (denoted by S) and a wear-out failure (denoted by W). By taking into account that the system fails when the first failure mode occurs, one can consider the device as a series system with two components: mode S and mode W. In this study, $n = 30$ devices are observed up to their failure or the end of the test. A total of $n = 8$ devices (26.67%) did not fail during the observation period. Half of the devices ($n = 15$) failed due to component S and 23.33% of the devices failed due to component W. Thus, component S has 50% of uncensored observations and 50% of right-censored observations, and component W has 23.33% of uncensored observations and 76.67% of right-censored observations.

To obtain posterior quantities related to the posterior distribution of $\boldsymbol{\theta}_j = (\beta_j, \eta_j, \mu_j)$, for $j = 1, 2$, from (3) through MCMC simulations, we discarded the first 10,000 as burn-in samples and used a jump size of 30 to avoid correlation



(a)



(b)

FIGURE 11. Mean (symbol) and standard deviation (bars) of the MAE of W3PM and BSNP for scenarios 5 and 6. W3PM indicates the posterior mean obtained by proposed model, and BSNP indicates the nonparametric estimation proposed by [16]. (a) Scenario 5. (b) Scenario 6.

problems, obtaining a sample of size $1,000$. The chains' convergence was monitored, and good convergence results were obtained.

Table 2 lists the posterior means and posterior standard deviation for the parameters of shape (β_j), scale (η_j), location (μ_j) and expected time of components' lifetimes, $E(X_j|\boldsymbol{\theta}_j) = \mu_j + \eta_j\Gamma(1 + (1/\beta_j))$, for $j = 1, 2$, in which it is the expected value of a three-parameter Weibull distributed random variable. The posterior means of the expected time of the component lifetimes are 554.22 thousand cycles for component S and 311.13 thousand cycles for component W. The posterior mean and the 95% highest posterior density (HPD) pointwise band of the reliability function are illustrated in figures 12a and 12b.

TABLE 1. Percentage of censored data for each component in each scenario.

Scenario	Component	Side of censoring Left	Right	Total
1	1	72.40%	5.40%	77.80%
	2	21.10%	33.30%	54.40%
	3	6.50%	61.30%	67.80%
2	1	37.40%	25.70%	63.10%
	2	48.00%	32.80%	80.80%
	3	14.60%	41.50%	56.10%
3	1	34.20%	21.30%	55.50%
	2	27.20%	49.10%	76.30%
	3	38.60%	29.60%	68.20%
4	1	40.00%	30.50%	70.50%
	2	32.40%	25.60%	58.00%
	3	27.60%	43.90%	71.50%
5	1	28.70%	51.40%	80.10%
	2	50.40%	18.90%	69.30%
	3	62.50%	27.50%	90.00%
	4	61.70%	19.10%	80.80%
	5	8.50%	71.30%	79.80%
6	1	52.60%	33.30%	85.90%
	2	24.30%	59.50%	83.80%
	3	17.90%	69.50%	87.40%
	4	23.00%	39.70%	62.70%
	5	64.20%	16.00%	80.20%

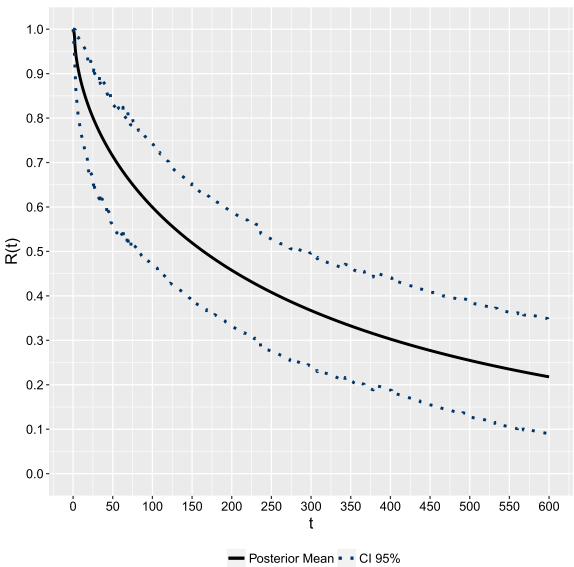
TABLE 2. Posterior mean and posterior standard deviation of proposed model parameters and mean time of component lifetimes for the device from a field-tracking data set.

	Component S	
	Posterior mean	Posterior standard deviation
β_1	0.62	0.14
η_1	311.93	98.47
μ_1	1.35	0.59
$E(X_1\|\boldsymbol{\theta}_1)$	554.22	410.68
	Component W	
	Posterior mean	Posterior standard deviation
β_2	4.24	1.31
η_2	341.64	37.42
μ_2	0.99	0.58
$E(X_2\|\boldsymbol{\theta}_2)$	311.13	32.01

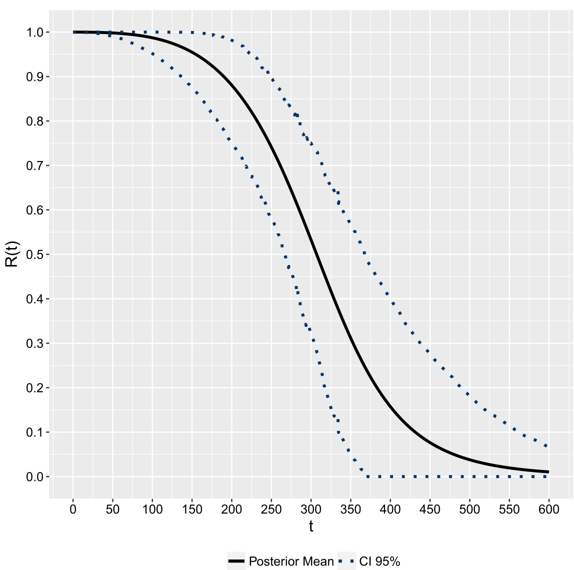
B. BOYS' FIRST USE OF MARIJUANA DATA SET

A social study is considered in which the proposed methodology can be suitably applied. The data are analyzed by [22]. In this study, $n = 191$ California high school boys were asked about their first use of marijuana. The answers were age in years if the responder did use and remember his age or "I never used it" (which are right-censored observations of the boys' current ages) or "I have used it but I cannot remember the exactly time for my first use of the drug" (which is a left-censored observation case) [22].

Reference [22] analyzed the data through Turnbull's estimator [23]. In their approach, boys who remember their ages when they first used the drug produced uncensored observations. Consider, for instance, a boy saying that he used the drug for the first time at 13 years old and, more specifically, that it happened when he was 13 years and 11 months old. He would be considered a subject with an



(a)



(b)

FIGURE 12. **Posterior mean and 95% HPD point-wise band (CI 95%) of reliability function for the lifetimes of components S and W. (a) Component S. (b) Component W.**

uncensored observation at the age of 13 by the Klein and Moeschberger analysis, even if his age of use was closer to 14 years. We believe that this kind of information should be considered an interval-censored observation at [13, 14), and his information would be properly taken as interval-censored in our likelihood, the second factor of the right-hand side of equation (1). In this way, all data are censored: either right-, left-, or interval-censored data.

To obtain posterior quantities related to the posterior distribution of $\boldsymbol{\theta} = (\beta, \eta, \mu)$ from (3) through MCMC simulations, we discarded the first 10,000 as burn-in samples and used a jump of size 20 to avoid correlation problems, obtaining a

TABLE 3. **Posterior mean and posterior standard deviation of proposed model parameters and mean time of first use of marijuana.**

	Posterior mean	Posterior standard deviation
β	2.4	0.37
η	6.19	0.58
μ	9.54	0.52
$E(X\|\boldsymbol{\theta})$	15.05	0.23

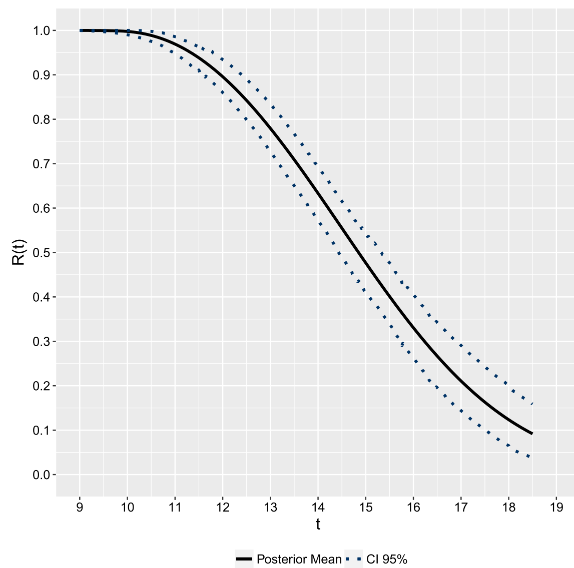


FIGURE 13. **Posterior mean and 95% HPD point-wise band of reliability function for time to first use of marijuana.**

sample size of 1,000. The chains' convergence was monitored for good convergence results to be obtained.

Table 3 lists the posterior means and posterior standard deviation for the parameters of shape (β), scale (η), location (μ) and expected time of first use of marijuana, $E(X|\boldsymbol{\theta}) = \mu + \eta\Gamma(1 + (1/\beta))$. The posterior mean of the expected time of first marijuana use is 15.05 years. The posterior mean and the 95% highest posterior density (HPD) point-wise band of the reliability function are illustrated in Figure 13.

V. FINAL REMARKS

A Bayesian Weibull model for component reliability was proposed. Identical distributions of component lifetimes were not imposed. The proposed methodology is said to be general because it can be used for any coherent system, from the simplest to the most complex designs. It is also appropriate for all kinds of censored data, including interval-censored, allowing it to be used in survival problems. In estimation processes, satisfactory results about the convergence of the MCMC method were obtained, and it was proved that the posterior is proper even when using prior distributions chosen from a family of non-informative prior distributions. We worked with the Bayesian Weibull model. However, it is quite simple to extend the work to other distributions or even to the pure likelihood approach [24].

Reference [16] also considered coherent systems, as we have here. However, the assumption of independent and identically distributed component lifetimes excludes the use of their method for most practical applications. On the other hand, their methodology does not require the choice of a parametric family of distributions. For positive random variables, we believe that the three-parameter Weibull family is a very rich family, since most real situations will have random aspects that can be represented by an component of the family.

Both methods were evaluated in scenarios with different distributions for the generation of component lifetimes, different percentages of censored data and different sample sizes. The observed information consists of the failure time of systems and the status of each component at the moment of each system failure. The simulation study showed excellent performance of the proposed estimator and that its advantage increases with the sample size. For the cases where the W3PM did not perform better, it was still very close to the performance of BSNP.

The practical relevance was assessed in two real data sets. The first is device from a field-tracking with two components in series, and the second is a social study in which interval-censored data appears. The proposed methodology can be suitably applied data sets. We also believe that for future work, our methodology can be used to perform reverse engineering. Using model selection Bayesian techniques, as in [25], that use a mixture of reliability estimates, one can choose one out of several alternative reliability systems.

APPENDIX

PROOF OF THEOREM 1

The proof for a series system was given by [17]. To prove the Theorem 1, we show that the condition is necessary for a parallel system. Then, we extend it to a series-parallel system and a parallel-series system. Finally, using the arguments for these results, we conclude with the condition for coherent systems.

The j-th sub-reliability function evaluated at time t is the probability that the system survives at least t and that the failure of the j-th component produced the system failure, that is, $G_j(t) = \Pr\Big\{(T > t) \cap (\delta = j)\Big\}$. Let $Q_j(t) = \Pr\Big\{(T \leq t) \cap (\delta = j)\Big\}$ be the j-th sub-distribution function evaluated at time t.

Let $R_{1,\ldots,m}(t_1, \ldots, t_m) = \Pr\Big\{\bigcap_{j=1}^{m}(X_j > t_j)\Big\}$ and $F_{1,\ldots,m}(t_1, \ldots, t_m) = \Pr\Big\{\bigcap_{j=1}^{m}(X_j \leq t_j)\Big\}$ be the joint reliability and the joint distribution functions, respectively, in which continuous partial derivatives with respect to all of its arguments are assumed.

The failure time of a parallel system with m components is $T = \max\{X_1, \ldots, X_m\}$. Reference [8] proved the following lemma.

Lemma 1: The derivative of $Q_j(t)$, $\mathrm{d}Q_j(t)/\,\mathrm{d}t$, is equal to the partial derivative of $F_{1,\ldots,m}(t_1, \ldots, t_m)$ at the jth component, evaluated at $t_1 = t_2 = \ldots = t_m = t$.

If the components' lifetimes are assumed to be mutually independent,

$$F_{1,\ldots,m}(t_1, \ldots, t_m) = \prod_{j=1}^{m} F_j(t_j). \tag{5}$$

Using the fact in (5) and the Lemma 1,

$$\frac{\mathrm{d}Q_j(t)}{\mathrm{d}t} = u_j(t)\prod_{j=1}^{m} F_j(t), \tag{6}$$

where u_j is the reversed hazard rate (RHR) of the jth component:

$$u_j(t) = \frac{f_j(t)}{F_j(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln(F_j(t)).$$

Note that

$$\begin{aligned} F_j(t) &= \exp\{-(-\ln(F_j(t)))\} \\ &= \exp\Big\{-[\ln(F_j(\infty)) - \ln(F_j(t))]\Big\} \\ &= \exp\Big\{-\int_t^{\infty}\frac{f_j(y)}{F_j(y)}\,\mathrm{d}y\Big\} \\ &= \exp\Big\{-\int_t^{\infty} u_j(y)\,\mathrm{d}y\Big\}. \end{aligned} \tag{7}$$

Letting $u(y) = \sum_{j=1}^{m} u_j(y)$, (6) becomes

$$\frac{\mathrm{d}Q_j(t)}{\mathrm{d}t} = u_j(t)\exp\Big\{-\int_t^{\infty} u(y)\,\mathrm{d}y\Big\}.$$

Lemma 1 indicates that any given joint distribution function uniquely determines the sub-distribution functions

$$Q_j(t) = \int_0^t \left[\left.\frac{\mathrm{d}F_{1,\ldots,m}(y_1, \ldots, y_m)}{\mathrm{d}y_j}\right|_{y_1=y_2=\ldots=y_m=y}\right]\mathrm{d}y.$$

Lemma 2 (For Parallel System): Let $X_1, \ldots, X_m$ be the failure time of the m components involved in a parallel system. Whatever the set of sub-distribution functions $Q_j(t)$, for $j = 1, \ldots, m$, there exists a system of marginal distribution functions, say, $F_j^*(t)$ for $j = 1, 2, \ldots, m$. This, combined with the assumption that the components' failure times are mutually independent, implies the sub-distribution functions $Q_j^*(t)$ that coincide with the given $Q_j(t)$.

Proof of Lemma 2: The proof consists in solving the equations

$$\frac{\mathrm{d}Q_j(t)}{\mathrm{d}t} = u_j^*(t)\exp\Big\{-\int_t^{\infty} u^*(y)\,\mathrm{d}y\Big\}, \quad j = 1, \ldots, m, \tag{8}$$

with respect to the $u_j^*(t)$, where $u^*(t) = \sum_{j=1}^{m} u_j^*(t)$. Taking now the sum for $j = 1, \ldots, m$, we obtain

$$\begin{aligned} \sum_{j=1}^{m}\frac{dQ_j(t)}{dt} &= u^*(t)\exp\Big\{-\int_t^{\infty} u^*(y)\,\mathrm{d}y\Big\} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\exp\Big\{-\int_t^{\infty} u^*(y)\,\mathrm{d}y\Big\}. \end{aligned}$$

Consequently,

$$\sum_{j=1}^{m} Q_j(t) = \exp\left\{ -\int_t^\infty u^*(y)\,\mathrm{d}y \right\},$$

which combined with (8), leads to

$$u_j^*(t) = \frac{\mathrm{d}Q_j(t)/\,\mathrm{d}t}{\sum_{j=1}^m Q_j(t)}.$$

Finally, (7) implies

$$F_j^*(t) = \exp\left\{ -\int_t^\infty \frac{\mathrm{d}Q_j(y)}{\sum_{j=1}^m Q_j(y)} \right\}. \tag{9}$$

The substitution of F_j^* and u_j^* into (6) will produce

$$\begin{aligned}
\frac{\mathrm{d}Q_j^*(t)}{\mathrm{d}t} &= u_j^*(t) \prod_{j=1}^m F_j^*(t) \\
&= \frac{\mathrm{d}Q_j(t)/\,\mathrm{d}t}{\sum_{j=1}^m Q_j(t)} \exp\left\{ -\int_t^\infty \frac{\sum_{j=1}^m \mathrm{d}Q_j(y)}{\sum_{j=1}^m Q_j(y)} \right\} \\
&= \frac{\mathrm{d}Q_j(t)/\,\mathrm{d}t}{\sum_{j=1}^m Q_j(t)} \exp\left\{ -\left[\log\left(\sum_{j=1}^m Q_j(\infty) \right) \right.\right. \\
&\qquad \left.\left. - \log\left(\sum_{j=1}^m Q_j(t) \right) \right] \right\} \\
&= \frac{\mathrm{d}Q_j(t)/\,\mathrm{d}t}{\sum_{j=1}^m Q_j(t)} \exp\left\{ \log\left(\sum_{j=1}^m Q_j(t) \right) \right\} \\
&= \frac{\mathrm{d}Q_j(t)}{\mathrm{d}t},
\end{aligned}$$

that is, the derivative of Q_j^* coincides with that of Q_j. ■

Now, let X_1, X_2, and X_3 be the lifetimes of three components of a series-parallel system (SPS) represented in Figure 2b. For component 1 in the SPS representation, we have that

$$\begin{aligned}
G_1(t) &= \Pr\left\{ (T > t) \cap (\delta = 1) \right\} \\
&= \Pr\left\{ (X_1 > t) \cap (Y > X_1) \right\},
\end{aligned}$$

where $Y = \max\{X_2, X_3\}$ and $T = \min\{X_1, \max\{X_2, X_3\}\}$.

Let $R_{1,Y}(t_1, y) = \Pr((X_1 > t_1) \cap (Y > y))$, and we have that

$$\frac{\mathrm{d}G_1(t)}{\mathrm{d}t} = \left. \frac{\mathrm{d}R_{1,Y}(t_1, y)}{\mathrm{d}t_1} \right|_{t_1=y=t}. \tag{10}$$

If X_1 and Y are assumed to be independent,

$$R_{1,Y}(t_1, y) = R_1(t_1) R_Y(y). \tag{11}$$

Using the facts in (11) and (10)

$$\frac{\mathrm{d}G_1(t)}{\mathrm{d}t} = -r_1(t) R_1(t) R_Y(t),$$

where

$$r_j(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln R_j(t)$$

and

$$R_j(t) = \exp\left\{ -\int_0^t r_j(x)\,\mathrm{d}x \right\}. \tag{12}$$

By the results of [17], we have that

$$r_1(t) = -\frac{G_1(t)}{G_1(t) + G_Y(t)},$$

in which $G_Y(t) = \Pr\{(Y > t) \cap (X_1 > Y)\}$ and

$$R_1(t) = \exp\left\{ \int_0^t \frac{\mathrm{d}G_1(x)}{G_1(x) + G_Y(x)} \right\}. \tag{13}$$

Since the system is in a series of component 1 with a maximum of components 2 and 3, we can use the result of [17] that whatever the set of sub-reliability functions $G_1(t)$ is, there exists marginal reliability functions, say, $R_1^*(t)$, that, combined with the assumption that component 1 failure time is independent of Y, implies the sub-reliability functions $G_1^*(t)$ that coincide with the given $G_1(t)$.

For component 2 in the SPS representation, we have that

$$\begin{aligned}
Q_2(t) &= \Pr\left\{ (T \le t) \cap (\delta = 2) \right\} \\
&= \Pr\left\{ (X_1 > X_2) \cap (X_2 \le t) \cap (X_3 < X_2) \right\}.
\end{aligned}$$

Lemma 3: For the system SPS in Figure 2b, the derivative of $Q_2(t)$, $\mathrm{d}Q_2(t)/\,\mathrm{d}t$ is given by

$$\begin{aligned}
&\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} \\
&= \left. \frac{\mathrm{d}F_{2,3}(t_2, t_3)}{\mathrm{d}t_2} \right|_{t_2=t_3=t} - \left. \frac{\mathrm{d}F_{1,2,3}(t_1, t_2, t_3)}{\mathrm{d}t_2} \right|_{t_1=t_2=t_3=t}.
\end{aligned}$$

Proof of Lemma 3: The difference

$$\begin{aligned}
&Q_2(t+h) - Q_2(t) \\
&= \Pr\left\{ (X_1 > X_2) \cap (t < X_2 \le t+h) \cap (X_3 < X_2) \right\}
\end{aligned} \tag{14}$$

has lower bound

$$\begin{aligned}
&\Pr\left\{ (X_1 > t+h^*) \cap (t < X_2 \le t+h^*) \cap (X_3 < t) \right\} \\
&= F_{1,2,3}(\infty, t+h, t) - F_{1,2,3}(\infty, t, t) \\
&\quad - [F_{1,2,3}(t+h^*, t+h, t) - F_{1,2,3}(t+h^*, t, t)],
\end{aligned} \tag{15}$$

and upper bound

$$\begin{aligned}
&\Pr\left\{ (X_1 > t) \cap (t < X_2 \le t+h^*) \cap (X_3 < t+h^*) \right\} \\
&= F_{1,2,3}(\infty, t+h, t+h^*) - F_{1,2,3}(\infty, t, t+h^*) \\
&\quad - [F_{1,2,3}(t, t+h, t+h^*) - F_{1,2,3}(t, t, t+h^*)].
\end{aligned} \tag{16}$$

By dividing (14), (15) and (16) by h and applying the limit as $h \to 0$, we obtain for all $h^* > 0$,

$$\left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=\infty,\\ t_2=t_3=t}} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=t+h^*,\\ t_2=t_3=t}}$$
$$\leq \frac{\mathrm{d}Q_2(t)}{\mathrm{d}t}$$
$$\leq \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=\infty,\\ t_2=t,\\ t_3=t+h^*}} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=t_2=t,\\ t_3=t+h^*}}.$$

Now, taking the limit $h^* \to 0$ produces the desired result

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}F_{2,3}(t_2,t_3)}{\mathrm{d}t_2}\right|_{t_2=t_3=t} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{t_1=t_2=t_3=t}. \quad (17)$$

Thus,

$$Q_2(t) = \int_0^t \left[\left.\frac{\mathrm{d}F_{2,3}(y_2,y_3)}{\mathrm{d}y_2}\right|_{y_2=y_3=y} - \left.\frac{\mathrm{d}F_{1,2,3}(y_1,y_2,y_3)}{\mathrm{d}y_2}\right|_{y_1=y_2=y_3=y}\right]\mathrm{d}y.$$

■

If the components' lifetimes are assumed to be mutually independent, using the fact in (5) and (17), we have that

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = u_2(t)\Big\{F_2(t)F_3(t)[1-F_1(t)]\Big\}. \quad (18)$$

Lemma 4 (For Component 2 of the SPS in Figure 2b): Let X_1, X_2, X_3 be the failure times of the 3 components involved in a series-parallel system given in Figure 2b. Whatever the set of sub-distribution functions $Q_2(t)$, there exists a system of marginal distribution functions, say, $F_2^*(t)$, that, combined with the assumption that the components' failure times are mutually independent, implies the sub-distribution functions $Q_2^*(t)$ coincide with the given $Q_2(t)$.

Proof of Lemma 4: The proof consists in solving the equations

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = u_2^*(t)\Big\{F_2^*(t)F_3(t)[1-F_1(t)]\Big\},$$

with respect to the $u_2^*(t)$.

By (7), we have that

$$F_2(t) = \exp\left\{-\int_t^\infty u_2(y)\,\mathrm{d}y\right\}.$$

We can write the integration as

$$\int_t^\infty u_2(y)\,\mathrm{d}y = \int_t^\infty \frac{\mathrm{d}F_2(x)}{F_2(x)} = \int_t^\infty \frac{[1-F_1(x)]F_3(x)\,\mathrm{d}F_2(x)}{[1-F_1(x)]F_3(x)F_2(x)}. \quad (19)$$

From (18), we have that $\mathrm{d}Q_2(x) = [1-F_1(x)]F_3(x)\,\mathrm{d}F_2(x)$. The system distribution function of the SPS system ($T = \min\{X_1, \max\{X_2, X_3\}\}$) is $F(t) = 1 - [1-F_1(t)][1 - F_2(t)F_3(t)]$. Thus, $F(t) - F_1(t) = [1-F_1(t)]F_2(t)F_3(t)$. In this way, we can write (19) as

$$\int_t^\infty u_2(y)\,\mathrm{d}y = \int_t^\infty \frac{\mathrm{d}Q_2(x)}{F(x)-F_1(x)} = \int_t^\infty \frac{\mathrm{d}Q_2(x)}{\sum_{j=1}^3 Q_j(x) - \Phi_s(Q_1,Q_2,Q_3,x)},$$

in which

$$\Phi_s(Q_1,Q_2,Q_3,t) = 1 - \exp\left\{\int_0^t \frac{-\,\mathrm{d}Q_1(t)}{1-\sum_{j=1}^3 Q_j(t)}\right\},$$

once $F(t) = \sum_{j=1}^3 Q_j(t)$ and $R_1(t) = 1 - F_1(t)$ can be written as

$$R_1(t) = \exp\left\{\int_0^t \frac{-\,\mathrm{d}Q_1(t)}{1-\sum_{j=1}^3 Q_j(t)}\right\},$$

what is obtained by (13) considering $Q_Y(t) = Q_2(t) + Q_3(t)$, under independence assumption.

Thus,

$$u_2^*(t) = \frac{\mathrm{d}Q_2(t)/\,\mathrm{d}t}{\sum_{j=1}^3\ Q_j(t) - \Phi_s(Q_1,Q_2,Q_3,t)}$$

and

$$F_2^*(t) = \exp\left\{-\int_t^\infty \frac{\mathrm{d}Q_2(x)}{\sum_{j=1}^3\ Q_j(x) - \Phi_s(Q_1,Q_2,Q_3,x)}\right\}.$$

The substitution of u_2^* into (18) will yield

$$\frac{\mathrm{d}Q_2^*(t)}{\mathrm{d}t} = u_2^*(t)\left\{\sum_{j=1}^3 Q_j(t) - \Phi_s(Q_1,Q_2,Q_3,t)\right\}$$
$$= \frac{\mathrm{d}Q_2(t)/\,\mathrm{d}t}{\sum_{j=1}^3\ Q_j(t) - \Phi_s(Q_1,Q_2,Q_3,t)} \times\left\{\sum_{j=1}^3 Q_j(t) - \Phi_s(Q_1,Q_2,Q_3,t)\right\}$$
$$= \frac{\mathrm{d}Q_2(t)}{\mathrm{d}t},$$

that is, the derivative of Q_2^* coincides with that of Q_2. ■

For component 3, we have that

$$Q_3(t) = \Pr\Big\{(T \leq t)\cap(\delta = 3)\Big\} = \Pr\Big\{(X_1 > X_3)\cap(X_2 < X_3)\cap(X_3 \leq t)\Big\}.$$

Thus, the proof for component 3 is analogous to that presented for component 2.

Now, let X_1, X_2 and X_3 be the lifetimes of three components of a parallel-series system (PSS) with representation in Figure 2a.

For component 1 in the PSS representation, we have that

$$Q_1(t) = \Pr\left\{(T \le t) \cap (\delta = 1)\right\} = \Pr\left\{(X_1 \le t) \cap (W < X_1)\right\},$$

where $W = \min\{X_2, X_3\}$ and $T = \max\{X_1, \min\{X_2, X_3\}\}$. Since the system is in parallel of component 1 with the minimum of components 2 and 3, we can consider the Lemma 2 for component 1.

For component 2 in the PSS representation, we have that

$$Q_2(t) = \Pr\left\{(T \le t) \cap (\delta = 2)\right\} = \Pr\left\{(X_1 < X_2) \cap (X_2 \le t) \cap (X_3 > X_2)\right\}.$$

Lemma 5: For the system PSS in Figure 2a, the derivative of $Q_2(t)$, $\mathrm{d}Q_2(t)/\,\mathrm{d}t$ is given by

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}F_{1,2}(t_1,t_2)}{\mathrm{d}t_2}\right|_{t_1=t_2=t} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{t_1=t_2=t_3=t}.$$

Proof of Lemma 5: The difference

$$Q_2(t+h) - Q_2(t) = \Pr\left\{(X_1 < X_2) \cap (t < X_2 \le t+h) \cap (X_3 > X_2)\right\} \tag{20}$$

has lower bound

$$\begin{aligned}\Pr&\left\{(X_1 < t) \cap (t < X_2 \le t+h^*) \cap (X_3 > t+h^*)\right\}\\ &= F_{1,2,3}(t, t+h, \infty) - F_{1,2,3}(t,t,\infty)\\ &\quad - [F_{1,2,3}(t,t+h,t+h^*) - F_{1,2,3}(t,t,t+h^*)],\end{aligned} \tag{21}$$

and upper bound

$$\begin{aligned}\Pr&\left\{(X_1 < t+h^*) \cap (t < X_2 \le t+h^*) \cap (X_3 > t)\right\}\\ &= F_{1,2,3}(t+h^*, t+h, \infty) - F_{1,2,3}(t+h^*,t,\infty)\\ &\quad - [F_{1,2,3}(t+h^*,t+h,t) - F_{1,2,3}(t+h^*,t,t)].\end{aligned} \tag{22}$$

By dividing (20), (21) and (22) by h and applying the limit as $h \to 0$, we obtain for all $h^* > 0$,

$$\begin{aligned}&\left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=t_2=t,\\ t_3=\infty}} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=t_2=t,\\ t_3=t+h^*}}\\ &\le \frac{\mathrm{d}Q_2(t)}{\mathrm{d}t}\\ &\le \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=t+h^*,\\ t_2=t,\\ t_3=\infty}} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{\substack{t_1=t+h^*,\\ t_2=t_3=t}}.\end{aligned}$$

Now, taking the limit $h^* \to 0$ produces the desired result

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}F_{1,2}(t_1,t_2)}{\mathrm{d}t_2}\right|_{t_1=t_2=t} - \left.\frac{\mathrm{d}F_{1,2,3}(t_1,t_2,t_3)}{\mathrm{d}t_2}\right|_{t_1=t_2=t_3=t}. \tag{23}$$

■

If the components' lifetimes are assumed to be mutually independent, using the fact in (5) and (23),

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = -r_2(t)\Big\{[1-F_2(t)]F_1(t)[1-F_3(t)]\Big\}, \tag{24}$$

Lemma 6 (For Component 2 of the PSS in Figure 2a): Let X_1, X_2, X_3 be the failure times of the 3 components involved in a parallel-series system given in Figure 2b. Whatever the set of sub-distribution functions $Q_2(t)$, there exists a system of marginal distribution functions, say, $F_2^*(t)$, that, combined with the assumption that the components' failure times are mutually independent, implies the sub-distribution functions $Q_2^*(t)$ coincide with the given $Q_2(t)$.

Proof of Lemma 6: The proof consists in solving the equations

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = -r_2^*(t)\Big\{[1-F_2^*(t)]F_1(t)[1-F_3(t)]\Big\},$$

with respect to the $r_2^*(t)$.

By (12), we have that

$$R_2(t) = 1 - F_2(t) = \exp\left\{-\int_0^t r_2(y)\,\mathrm{d}y\right\}.$$

We can write the integration as

$$\begin{aligned}\int_0^t r_2(y)\,\mathrm{d}y &= \int_0^t -\frac{\mathrm{d}[1-F_2(x)]}{[1-F_2(x)]}\\ &= \int_0^t -\frac{F_1(x)[1-F_3(x)]\,\mathrm{d}[1-F_2(x)]}{F_1(x)[1-F_3(x)][1-F_2(x)]}\end{aligned} \tag{25}$$

From (24), we have that $\mathrm{d}Q_2(x) = F_1(x)[1-F_3(x)]\,\mathrm{d}[1-F_2(x)]$. The system distribution function of the PSS system is $F(t) = F_1(t)\{1-[1-F_2(t)][1-F_3(t)]\}$. Thus, $F_1(t)-F(t) = F_1(t)[1-F_2(t)][1-F_3(t)]$. In this way, we can write (25) as

$$\begin{aligned}\int_0^t r_2(y)\,\mathrm{d}y &= \int_t^\infty -\frac{\mathrm{d}Q_2(x)}{F_1(x)-F(x)}\\ &= \int_0^t -\frac{\mathrm{d}Q_2(x)}{\Phi_p(Q_1,Q_2,Q_3,x) - \sum_{j=1}^3 Q_j(x)},\end{aligned}$$

in which

$$\Phi_p(Q_1,Q_2,Q_3,t) = \exp\left\{\int_t^\infty -\frac{\mathrm{d}Q_1(t)}{\sum_{j=1}^3 Q_j(t)}\right\},$$

once $F_1(t)$ can be written as (9) and $F(t) = \sum_{j=1}^3 Q_j(t)$.

We have that

$$r_2^*(t) = -\frac{\mathrm{d}Q_2(x)}{\Phi_p(Q_1,Q_2,Q_3,x) - \sum_{j=1}^3 Q_j(x)}$$

and

$$F_2^*(t) = 1 - \exp\left\{-\int_0^t \frac{\mathrm{d}Q_2(x)}{\Phi_p(Q_1, Q_2, Q_3, x) - \sum_{j=1}^{3} Q_j(x)}\right\}.$$

The substitution of r_2^* into (24) will produce the derivative of Q_2^*, which coincides with that of Q_2. ■

For component 3, we have that

$$\begin{aligned} Q_3(t) &= \Pr\left\{(T \le t) \cap (\delta = 3)\right\} \\ &= \Pr\left\{(X_1 < X_3) \cap (X_2 > X_3) \cap (X_3 \le t)\right\}. \end{aligned}$$

Thus, the proof for component 3 is analogous to that presented for component 2.

Proof of Theorem 1: Considering the interest in directly accessing the marginal distribution of each component in a coherent system, we have that: 1) the series and parallel systems are particular cases of the class of coherent systems; 2) any coherent system can be written as a combination of PSS and SPS. Therefore, the lemmas 1 to 6, together with the results from [17], show that without the hypothesis that the components' survival times are mutually independent, the model of marginal survival times is unidentifiable: the set of sub-reliability functions (sub-distribution functions) is consistent with an infinity of joint reliability functions (joint distribution functions) for coherent systems. ■

PROOF OF THEOREM 2

Proof: We have to show that

$$\int_0^{\min\{t\}} \int_0^\infty \int_0^\infty \pi(\beta_j, \eta_j, \mu_j \mid l_j, u_j)\, \mathrm{d}\beta_j\, \mathrm{d}\eta_j\, \mathrm{d}\mu_j < \infty,$$

where

$$\begin{aligned} &\pi(\beta_j, \eta_j, \mu_j \mid l_j, u_j) \propto \pi(\beta_j, \eta_j, \mu_j) \\ &\times \prod_{i=1}^{n} \left\{ \left(\frac{l_{ji}-\mu_j}{\eta_j}\right)^{\beta_j - 1} \frac{\beta_j}{\eta_j} \exp\left[-\left(\frac{l_{ji}-\mu_j}{\eta_j}\right)^{\beta_j}\right] \right\}^{I_{\{l_{ji}=u_{ji}\}}} \\ &\times \left\{ \exp\left[-\left(\frac{l_{ji}-\mu_j}{\eta_j}\right)^{\beta_j}\right] \right. \\ &\left. - \exp\left[-\left(\frac{u_{ji}-\mu_j}{\eta_j}\right)^{\beta_j}\right] \right\}^{1-I_{\{l_{ji}=u_{ji}\}}}. \end{aligned}$$

Because this proof works for all j, we will omit the j index.

For $n = 1$ and $l_1 = u_1$,

$$\begin{aligned} &\int_0^{l_1} \int_0^\infty \int_0^\infty \frac{1}{\eta\beta^b} \left(\frac{l_1-\mu}{\eta}\right)^{\beta-1} \frac{\beta}{\eta} \\ &\qquad \times \exp\left[-\left(\frac{l_1-\mu}{\eta}\right)^{\beta}\right] \mathrm{d}\beta\, \mathrm{d}\eta\, \mathrm{d}\mu \\ &= \int_0^{l_1} \int_0^\infty \frac{1}{\beta^b} \int_0^\infty \frac{\beta(l_1-\mu)^{\beta-1}}{\eta^{\beta+1}} \\ &\qquad \times \exp\left\{-\left(\frac{l_1-\mu}{\eta}\right)^{\beta}\right\} \mathrm{d}\eta\, \mathrm{d}\beta\, \mathrm{d}\mu. \qquad (26) \end{aligned}$$

Let X be a random variable that, given α and γ, follows an inverse gamma distribution. Its density function is expressed as

$$f(x \mid \alpha, \gamma) = \frac{\gamma^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left\{-\frac{\gamma}{x}\right\}, \quad \alpha > 0 \text{ and } \gamma > 0.$$

Consider the variable change: $[(l_1 - \mu)/\eta]^\beta = \gamma/x$, from which it follows that $(l_1-\mu)^\beta\, \mathrm{d}x = \gamma\beta\eta^{\beta-1}\, \mathrm{d}\eta$, so the integral expression in (26) can be written as

$$\begin{aligned} &\int_0^{l_1} \int_0^\infty \frac{1}{\beta^b} \int_0^\infty \left(\frac{\gamma}{x}\right)^2 \frac{1}{(l_1-\mu)\gamma} \exp\left\{-\frac{\gamma}{x}\right\} \mathrm{d}x\, \mathrm{d}\beta\, \mathrm{d}\mu \\ &= \int_0^{l_1} \int_0^\infty \frac{1}{\beta^b (l_1-\mu)} \int_0^\infty \gamma x^{-2} \exp\left\{-\frac{\gamma}{x}\right\} \mathrm{d}x\, \mathrm{d}\beta\, \mathrm{d}\mu \\ &= \int_0^{l_1} \int_0^\infty \frac{1}{\beta^b (l_1-\mu)} \mathrm{d}\beta\, \mathrm{d}\mu \\ &= \infty. \end{aligned}$$

In summary, for $n = 1$ and $l_1 = u_1$,

$$\int_0^{l_1} \int_0^\infty \int_0^\infty \pi(\beta, \eta, \mu \mid l_1, u_1)\, \mathrm{d}\beta\, \mathrm{d}\eta\, \mathrm{d}\mu = \infty.$$

Consider that for a sample of size n, $n > 1$, data are observed such that $l_i = u_i$, for $i = 1, \ldots, n_f$ and $l_i \ne u_i$ for $i = n_{f+1}, \ldots, n$.

$$\begin{aligned} &\pi(\beta, \eta, \mu \mid l, u) \propto \pi(\beta, \eta, \mu) \\ &\times \prod_{i=1}^{n_f} \left[\left(\frac{l_i-\mu}{\eta}\right)^{\beta-1} \frac{\beta}{\eta} \exp\left\{-\left(\frac{(l_i-\mu)}{\eta}\right)^{\beta}\right\} \right] \\ &\times \prod_{i=n_f+1}^{n} \left[\exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\} - \exp\left\{-\left(\frac{u_i-\mu}{\eta}\right)^{\beta}\right\} \right], \end{aligned}$$

where $t_l = (t_{l1}, \ldots, t_{ln})$, for $l = 1, 2$.

Because $l_i < u_i$, for all $i = 1, \ldots, n$, we have that

$$\exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\} > \exp\left\{-\left(\frac{u_i-\mu}{\eta}\right)^{\beta}\right\}.$$

In this way,

$$\begin{aligned} &\pi(\beta, \eta, \mu \mid l, u) \\ &\propto \pi(\beta, \eta, \mu) \prod_{i=1}^{n_f} \left[\left(\frac{l_i-\mu}{\eta}\right)^{\beta-1} \frac{\beta}{\eta} \exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\} \right] \\ &\quad \times \prod_{i=n_f+1}^{n} \left[\exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\} - \exp\left\{-\left(\frac{u_i-\mu}{\eta}\right)^{\beta}\right\} \right] \\ &< \pi(\beta, \eta, \mu) \prod_{i=1}^{n_f} \left[\left(\frac{l_i-\mu}{\eta}\right)^{\beta-1} \frac{\beta}{\eta} \exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\} \right] \\ &\quad \times \prod_{i=n_f+1}^{n} \left[\exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\} \right]. \end{aligned}$$

Thus, it is necessary only to evaluate the upper bound, that is,

$$\int_0^{\min\{t\}}\int_0^{\infty}\int_0^{\infty}\pi(\beta,\eta,\mu)\times\prod_{i=1}^{n_f}\left[\left(\frac{l_i-\mu}{\eta}\right)^{\beta-1}\frac{\beta}{\eta}\exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\}\right]\times\prod_{i=n_f+1}^{n}\left[\exp\left\{-\left(\frac{l_i-\mu}{\eta}\right)^{\beta}\right\}\right]\,\mathrm{d}\beta\,\mathrm{d}\eta\,\mathrm{d}\mu<\infty. \quad (27)$$

Let $t_{mi}=l_i-\mu$ and consider first the integrals in β and η, that is,

$$\begin{aligned}I&=\int_0^{\infty}\int_0^{\infty}\pi(\beta,\eta,\mu)\prod_{i=1}^{n_f}\left[\left(\frac{t_{mi}}{\eta}\right)^{\beta-1}\frac{\beta}{\eta}\times\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\prod_{i=n_f+1}^{n}\left[\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\mathrm{d}\beta\,\mathrm{d}\eta\\&=\int_0^{\infty}\int_0^{\infty}\frac{1}{\beta^b\eta}\prod_{i=1}^{n_f}\left[\left(\frac{t_{mi}}{\eta}\right)^{\beta-1}\frac{\beta}{\eta}\right]\times\prod_{i=1}^{n}\left[\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\mathrm{d}\beta\,\mathrm{d}\eta\\&=\int_0^{\infty}\int_0^{\infty}\frac{1}{\beta^b\eta}\frac{\beta^{n_f}}{\eta^{\beta n_f}}\prod_{i=1}^{n_f}\left[t_{mi}^{\beta-1}\right]\times\exp\left\{-\sum_{i=1}^{n}\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\mathrm{d}\beta\,\mathrm{d}\eta. \quad (28)\end{aligned}$$

Consider again the variable change: $\left(\sum_{i=1}^{n}t_{mi}^{\beta}/\eta^{\beta}\right)=\gamma/x$, from which it follows that $\sum_{i=1}^{n}t_{mi}^{\beta}\,\mathrm{d}x=\gamma\beta\eta^{\beta-1}\,\mathrm{d}\eta$. Thus, we can write the expression in (28) as

$$\begin{aligned}&\int_0^{\infty}\frac{\beta^{-b}}{\left[\sum_{i=1}^{n}t_{mi}^{\beta}\right]^{n_f}}\prod_{i=1}^{n_f}\left[t_{mi}^{\beta-1}\right]\times\int_0^{\infty}\beta^{n_f-1}\gamma^{n_f}x^{-n_f-1}\exp\left\{-\frac{\gamma}{x}\right\}\mathrm{d}x\,\mathrm{d}\beta\\&=\int_0^{\infty}\frac{\beta^{n_f-b-1}}{\left[\sum_{i=1}^{n}t_{mi}^{\beta}\right]^{n_f}}\prod_{i=1}^{n_f}\left[t_{mi}^{\beta-1}\right]\Gamma(n_f)\times\int_0^{\infty}\frac{\gamma^{n_f}}{\Gamma(n_f)}x^{-n_f-1}\exp\left\{-\frac{\gamma}{x}\right\}\mathrm{d}x\,\mathrm{d}\beta\\&=\int_0^{\infty}\frac{\beta^{n_f-b-1}}{\left[\sum_{i=1}^{n}t_{mi}^{\beta}\right]^{n_f}}\prod_{i=1}^{n_f}\left[t_{mi}^{\beta-1}\right]\Gamma(n_f)\,\mathrm{d}\beta.\end{aligned}$$

Let c be a real positive number such that $c>\prod_{i=1}^{n_f}t_{mi}^{-1}$, and consider also t_{1v} such that $t_{1v}<\max(t_{m1},\ldots,t_{mn_f})$. In this way, for all $b>0$, we have that

$$\begin{aligned}&\int_0^{\infty}\beta^{n_f-b-1}\Gamma(n_f)\frac{\prod_{i=1}^{n_f}t_{mi}^{\beta-1}}{\left[\sum_{i=1}^{n}t_{mi}^{\beta}\right]^{n_f}}\,\mathrm{d}\beta\\&<\int_0^{\infty}c\beta^{n_f-b-1}\Gamma(n_f)\frac{\prod_{i=1}^{n_f}t_{mi}^{\beta}}{\left[\sum_{i=1}^{n}t_{mi}^{\beta}\right]^{n_f}}\,\mathrm{d}\beta\\&<\int_0^{\infty}c\beta^{n_f-b-1}\Gamma(n_f)\frac{\prod_{i=1}^{n_f}t_{mi}^{\beta}}{\left[\sum_{i=1}^{n_f}t_{mi}^{\beta}\right]^{n_f}}\,\mathrm{d}\beta\\&<c\Gamma(n_f)\int_0^{\infty}\beta^{n_f-b-1}\left[\frac{t_{1v}}{\max(t_{m1},\ldots,t_{mn_f})}\right]^{\beta}\mathrm{d}\beta. \quad (29)\end{aligned}$$

Let $h=t_{1v}/\max(t_{m1},\ldots,t_{mn_f})$. We can write the last expression in (29) as

$$c\Gamma(n_f)\int_0^{\infty}\beta^{n_f-b-1}h^{\beta}\,\mathrm{d}\beta=c\Gamma(n_f)\int_0^{\infty}\beta^{n_f-b-1}\exp\{\beta\ln(h)\}\,\mathrm{d}\beta. \quad (30)$$

Considering the variable change $v=-\beta\ln(h)$, we have that $\mathrm{d}v=-\ln(h)\,\mathrm{d}\beta$, and (30) can be expressed as

$$c\Gamma(n_f)\int_0^{\infty}\left(\frac{1}{|\ln(h)|}\right)^{n_f-b}v^{n_f-b-1}\exp\{-v\}\,\mathrm{d}v,$$

since $h<1$. Let $c_2=c\Gamma(n_f)\left(\frac{1}{|\ln(h)|}\right)^{n_f-b}$. In this way,

$$c_2\int_0^{\infty}v^{n_f-b-1}\exp\{-v\}\,\mathrm{d}t<c_2\int_0^{\infty}v^{z-1}\exp\{-v\}\,\mathrm{d}v<\infty, \quad (31)$$

where z is the smallest positive integer larger than n_f-b.

The result in (31) is valid, since for all positive integer a,

$$\Gamma(a)=\int_0^{\infty}v^{a-1}\exp\{-v\}\,\mathrm{d}v<\infty.$$

In this way, we have that

$$I=\int_0^{\infty}\int_0^{\infty}\pi(\beta,\eta,\mu)\times\prod_{i=1}^{n_f}\left[\left(\frac{t_{mi}}{\eta}\right)^{\beta-1}\frac{\beta}{\eta}\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\times\prod_{i=n_f+1}^{n}\left[\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\mathrm{d}\beta\,\mathrm{d}\eta<\infty \quad (32)$$

Returning to equation (27) and considering (32), we finally have that

$$\int_0^{\min\{t\}}\int_0^{\infty}\int_0^{\infty}\pi(\beta,\eta,\mu)\times\prod_{i=1}^{n_f}\left[\left(\frac{t_{mi}}{\eta}\right)^{\beta-1}\frac{\beta}{\eta}\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\times\prod_{i=n_f+1}^{n}\left[\exp\left\{-\left(\frac{t_{mi}}{\eta}\right)^{\beta}\right\}\right]\mathrm{d}\beta\,\mathrm{d}\eta\,\mathrm{d}\mu<\infty.$$

■

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