Borrower: IAX

Lending String: *FDA,NOC,WAU,WYU,CIN

Patron: Chervinko, M

Journal Title: Test.

Volume: 3 **Issue:** 3
Month/Year: 1994 **Pages:** 181-193

Article Author:

Article Title: Motivation for the use of discrete distributions in quality assurance. Irony, T.

Imprint: Madrid, Spain ; Sociedad Española de Est

ILL Number: 38773865

Call #: 510.5T342

Location: Dirac

Mail
Charge
Maxcost: $20IFM

Shipping Address:
Library of the Health Sciences-Borrowing
The University of Illinois at Chicago
1750 W. Polk St.
Chicago, IL 60612

Fax: (312) 996-1899
Ariel: 128.248.46.209
Odyssey:

__ **Saved on "P" Drive**
__ **NOS** (Not on Shelf)
__ **NFAC** (Not Found as Cited)

PROCESSED BY ____________________

Test (1994)
Vol. 3, *No.* 2, *pp.* 181–193

Motivation for the use of discrete distributions in quality assurance

TELBA Z. IRONY* and CARLOS A. B. PEREIRA**

**Dept. of Operations Research, The George Washington Univ., Washington, DC, 20052, U.S.A.*

***Dept. de Estatística - Inst. de Matemática e Estatística da Universidade de São Paulo, Caixa Postal 20.570 - Ag. Jd. Paulistano, São Paulo, Brazil.*

SUMMARY

By considering a finite population of N items and S defects, and observing the way defects should be distributed among the items we provide an interesting motivation to the binomial, negative binomial (geometric) and Poisson distributions for the number of defects in a sample from a production line. The idea is to find out, from physical considerations about the production process, which configurations of defects in items are equally likely. A uniform distribution is assessed on the space generated by these configurations. Then, a distribution for a finite, and subsequently for an infinite population of items is derived.

Keywords: DISCRETE DISTRIBUTIONS; EXCHANGEABILITY; FINITE POPULATIONS; PRINCIPLE OF INDIFFERENCE; MAJORIZATION; PRODUCTION PROCESS.

1. INTRODUCTION

Statisticians and researchers have, over a period of time, become accustomed to the use of "standard" probability distributions that fit "reasonably well" a particular process under analysis. In many cases the choice of the distribution is motivated by tradition or mathematical tractability, rather than theoretical justification. Herein we examine the justification

Received May 94; Revised October 94.

for the use of various discrete distributions in the area of quality assurance. Rather than mathematical convenience, the basis of the approach taken here is to motivate the probability distributions of the number of defects in a sample of items, based on physical characteristics underlying the production processes, and on the knowledge of the quality analyst in charge of the system. Starting with a finite population of units and defects and the judgment of exchangeability for units with respect to "receiving the defects", we derive the appropriate probability distributions to be used in quality assurance. Such distributions will be more realistic and this should translate into more accurate inferences. Barlow and Mendel (1992), and Hayakawa (1994) used a similar approach to develop appropriate probabilistic models for aging and lifetimes.

It has been standard practice to use de binomial distribution for attribute data and the Poisson distribution for count data. By using our approach, it can be shown that although the binomial model appears to be reasonable for modeling attribute data, the use of the Poisson distribution may not adequately model count data. In some cases, the negative binomial model is the most adequate.

2. THE PRINCIPLE OF INDIFFERENCE

The basic idea is to start with a finite population of N items and S defects and to think of all possible distinct ways in which the defects could be distributed among the items, based on physical considerations about the process that generates the items and defects. Next, also based on the analyst's knowlegde about the physics underlying the process, an attempt is made to identify which configurations of defects in items are equally likely. In other words, the analyst's knowledge about the process will determine a finite space of configurations of defects in items in which a uniform distribution should be specified. To assess a uniform distribution over such a space is to be indifferent among all possible configurations. This idea is based on the *principle of indiference* as specified in Mendel (1989). According to him, by saying that she is indifferent to bet on any configuration of defects in items, the analyst is saying that she considers any configuration to be equally likely, that is, she is specifying a uniform distribution on the set of all possible configurations. Typically, it is much easier to assess and to agree upon a uniform distribution than upon any other distribution.

Once this finite space has been indentified, the distributional form for the random quantity of interest (the number of defects in a sample of size n) is derived by using counting techniques. In this way, "*finite versions*" of the Poisson, negative binomial (geometric) and binomial distributions are derived. The traditional (infinite) versions of these distributions are then obtained as limits of the distributions for the finite populations of defects and items, as the populations increase without bound.

3. THE POISSON DISTRIBUTION

It is possible for a product to have a few minor defects without the entire product being classified as defective, as noted in Montgomery (1991). Defects may occur as blemishes in a bolt of fabric, flaws in a blade of steel, bubbles in the coating of a product, broken rivets in an aircraft wing, and so on.

Given the above, assume a population of N items and S defects, all distinguishable, and suppose that the quality analyst believes that defect j ($j = 1, \ldots, S$) will be located in item i ($i = 1, \ldots, N$) with probability N^{-1}. This means that all N items have the same chance N^{-1} of receiving defect j.

One could think of a number of production scenarios that would be consistent with this judgement. For instance:

(a) N aircraft wings will be assembled in an assembly line that uses rivets coming from a large lot containing S defective rivets. If the rivets are randomly distributed among the wings, it is reasonable to believe that a defective rivet j will be assembled in wing i with probability N^{-1}.

(b) A weaving machine produces fabric. The production starts in the morning and lasts the whole day. This results in a continuous length of cloth that is cut at the end of the day into N pieces of the same length to be rolled onto N bolts. The analyst in charge of the production knows that the weaving machine produces small blemishes in the fabric. Moreover, she believes that:

1. The process by which the blemishes are generated is stable (stationary), that is, the distribution of the number of blemishes which occur in any piece of cloth depends only on the length of the piece.
2. The number of blemishes which occur in different (disjoint) pieces of cloth are independent.

3. Blemishes never occur simultaneously; there is always some measurable interval between any pair.

It is worth noting that there are many situations in which these conditions are not met. If the fabric is weaved from threads supplied in batches, the average number of blemishes produced by the machine may depend on the kind of thread and consequently vary from batch to batch. In that case the process will not have stationary increments. Sometimes the occurence of a blemish will increase (as it will be seen in section 4) or decrease the probability of occurence of another in the proximity. If this is true the increments will not be independent.

Whenever all three conditions are met, the analyst is dealing with a Poisson process. If so, it is a well known result that, given that blemish j was generated during the day, the probability that it was located in bolt i is N^{-1}, $i = 1, \ldots, N$.

Now, define the variable x_{ij} as:

$$x_{ij} = \begin{cases} 1 & \text{if defect } j \text{ is in item } i; \text{ for } i = 1, 2, \ldots N \\ & \qquad\qquad\qquad\qquad \text{and } j = 1, 2, \ldots, S \\ 0 & \text{otherwise.} \end{cases}$$

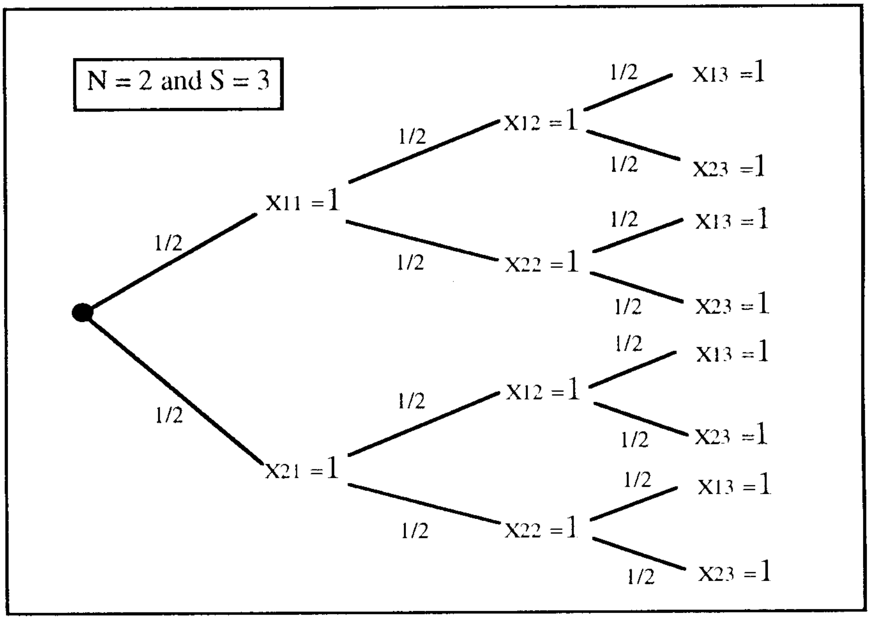
If the S defects and the N items are all distinguishable, there will be N^S possible configurations of defects in items (see Feller (1957), chapter II). If the probability that defect j will be located in item i is N^{-1}, then all the configurations are equally likely, that is, the probability of each of them is $(N^S)^{-1}$. In this case, we can write the conditional probability of the x_{ij} given N and S as:

$$p(x_{11}, \ldots, x_{1S}, x_{21}, \ldots, x_{2S}, \ldots, x_{N1}, \ldots, x_{NS} \mid N, S) = \frac{1}{N^S} .$$

In other words, the analyst who believes that the probability of defect j being located in item i is N^{-1}, is indifferent among all N^S possible configurations of defects in items (principle of indifference). Figure 1 illustrates the case where $N = 2$ and $S = 3$.

If $y_i = \sum_{j=1}^{S} x_{ij}$ denotes the number of defects in the i-th item, the joint conditional (on N and S) probability distribution that the items $1, 2, \ldots, N$ will have $y_1, y_2, \ldots, y_N$ defects respectively (where $\sum_{i=1}^{N} y_i = S$) is:

$$p(y_1, \ldots, y_N \mid N, S) = \binom{S}{y_1 \ldots y_N} \frac{1}{N^S} I\left(\sum_{i=1}^{N} y_i = S\right)$$



All branches of the tree have the same probability: 1/8.

Figure 1. *Example, with $N = 2$ and $S = 3$.*

where $I(A)$ is the indicator function, assuming the value 1 when A is true and the value 0 otherwise. $\binom{S}{y_1 \cdots y_N}$ is the number of ways of distributing the S defects in such a way that the first item has y_1 defects, the second item has y_2 defects, and so on. The above expression indicates that more diverse allocations of defects in items will be more probable than less diverse allocations. In other words, this is a case where it is more likely that the S defects are evenly spread among the N items. In terms of majorization, it is said that $p(y_1, \ldots, y_N \mid N, S)$ is Schur-concave (see Marshall and Olkin, 1979). As an example of the above consider the case of $N = 2$ and $S = 3$ depicted in Figure 1. Although the number of configurations is $2^3 = 8$, the number of ways in which item 1 has 2 defects and item 2 has 1 defect is $\binom{3}{2\ 1} = 3$.

Sampling distribution

Now, suppose we collect a sample of size n. Let $p(x|n, N, S)$ be the probability of having x defects in the sample. If we do not care about which defects are in which items, then x defects can be chosen in $\binom{S}{x}$ ways from the total of S defects. The remaining $(S - x)$ defects can be placed in the remaining $(N - n)$ items in $(N - n)^{S-x}$ ways. Moreover, there will be n^x different configurations of x defects in n items, each of them with probability $\frac{1}{N^S}$. Consequently:

$$\begin{aligned} p(x \mid n, N, S) &= \binom{S}{x} n^x \left(\frac{1}{N}\right)^S (N-n)^{S-x} \\ &= \binom{S}{x} \left(\frac{n}{N}\right)^x \left(1 - \frac{n}{N}\right)^{S-x} . \end{aligned}$$

This is the "*finite version*" of the Poisson distribution (a binomial distribution with parameters S and $\frac{n}{N}$). If the population of items and defects increases without bound, that is, if $N \to \infty$ and $S \to \infty$ so that $\frac{S}{N} = \lambda$ is bounded, we may use the Poisson approximation for the binomial. Then

$$p(x|n, \lambda) = \frac{e^{-n\lambda}(n\lambda)^x}{x!}$$

where λ is the average number of defects per item.

4. THE NEGATIVE BINOMIAL (GEOMETRIC) DISTRIBUTION

Here, we also deal with products that might have a number of minor defects. In line with this, we consider once more a population of N items with a total of S defects. However, in this case, the quality analyst belives that defect j ($j = 1, \ldots, S$) will be located in item i with probability p_i ($i = 1, \ldots, N$ and $\sum_{i=1}^{N} p_i = 1$), rather than $\frac{1}{N}$. In other words, the chance with which defect j will be placed in item 1 depends on the item i. Let y_i be the number of defects in item i. If $\boldsymbol{p} = (p_1, \ldots, p_N)$ were known, the conditional joint distribution of $\boldsymbol{y} = (y_1, \ldots, y_N)$ given $\boldsymbol{p}$,

N and S would be given by:

$$p(\boldsymbol{y} \mid \boldsymbol{p}, N, S) = \binom{S}{y_1 \cdots y_N} \prod_{i=1}^{N} p_i^{y_i} \quad \text{for } y_i \geq 0,\ p_i \geq 0,$$
$$\sum_{i=1}^{N} y_i = S \text{ and } \sum_{i=1}^{N} p_i = 1.$$

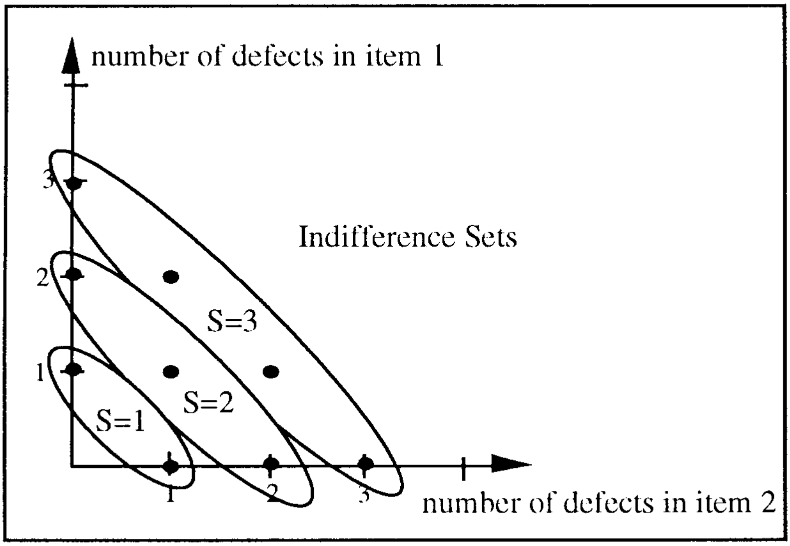
Suppose that, with no further information, the analyst assesses a uniform distribution to the vector $\boldsymbol{p}$ over the simplex

$$\mathcal{S} = \{\boldsymbol{p} = (p_1, \ldots, p_N) : p_i \geq 0 \ \text{ for } i = 1, \ldots, N \ \text{ and } \ \sum_{i=1}^{N} p_i = 1\},$$

that is, Dirichlet distribution with parameters $a_1 = \cdots = a_N = 1$. In other words, $f(\boldsymbol{p}|N) = (N-1)!$ for $\boldsymbol{p} \in \mathcal{S}$. Hence, considering the average of $p(\boldsymbol{y}|\boldsymbol{p}, N, S)$ over the simplex $\mathcal{S}$, we obtain the unconditional distribution of the vector $\boldsymbol{y}$:

$$\begin{aligned} p(\boldsymbol{y}|N, S) &= \int_{\mathcal{S}} \cdots \int p(\boldsymbol{y}|\boldsymbol{p}, N, S) f(\boldsymbol{p}|N) \mathrm{d}\boldsymbol{p} \\ &= \int_{\mathcal{S}} \cdots \int \binom{S}{y_1 \ldots y_N} \prod_{i=1}^{N} p_i^{y_i} (N-1)! \mathrm{d}\boldsymbol{p} \\ &= \frac{1}{\binom{N+S-1}{S}}. \end{aligned}$$

In fact, if the S defects are viewed as the same in severity (i.e., they are not distinguishable) and if we are interested only in the number of defects in each of the N items, it is possible ot obtain $\binom{N+S-1}{S}$ different configurations (see Feller, 1957, chapter II). Under the scenario just described, all configurations will be equally likely. This means that, if the analyst believes that a production process could be characterized as above, she should be indifferent to bet on any of such configurations (principle of indifference). Figure 2 illustrates the indifference sets for $N = 2$ and $S = 1, 2, 3$. Note from this figure, that more diverse allocations of defects to items (even spreading of defects over the items)



All points in each indifference set are equally likely.
For S=1, each point has probability 1/2.
For S=2, each point has probability 1/3.
For S=3, each point has probability 1/4.

Figure 2. *Indifference sets.*

are just as likely as less diverse allocations of defects to items (uneven spreading of defects over the items). In other words, $p(\boldsymbol{y}|N, S)$ is Schur-constant (see Marshall and Olkin, 1979).

A realistic situation that could result in an assessment like this would be the case in which the S defects were actually proceeding from many different sources, each having its own characteristic vector $\boldsymbol{p}$, which gives rise to a uniform spread over all possible $\boldsymbol{p}$ vectors.

An intuitive feeling about the type of production process that would generate such a space of equally likely configurations arises when we think of a process in which the defects are sequentially assigned to the items. In this case, we may compute the conditional probability that the $(k+1)$-th defect is placed in item i given that this item already has d defects. This can be written as:

$$p(x_{i,k+1} = 1 \,|D_i) = \frac{p(x_{i,k+1} = 1,\ D_i)}{p(D_i)}\ ,$$

where $D_i = \{\ d$ defects out of k were assigned to item $i\}$.

Recall that, if $\boldsymbol{p}$ has a uniform distribution over the simplex $\mathcal{S}$, then the marginal distribution of p_i, $f(p_i)$, $i = 1, \ldots, N$, is Beta with parameters 1 and $N - 1$.

$$f(p_i) = \begin{cases} (N-1)(1-p_i)^{N-2} & \text{for } 0 \leq p_i \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence:

$$p(x_{i,k+1} = 1 \mid D_i) = \frac{\int_0^1 p(x_{i,k+1} = 1, D_i \mid p_i) f(p_i) \mathrm{d}p_i}{\int_0^1 p(D_i \mid p_i) f(p_i) \mathrm{d}p_i} =$$

$$\frac{\int_0^1 p_i \binom{k}{d} p_i^d (1-p_i)^{k-d} f(p_i) \mathrm{d}p_i}{\int_0^1 \binom{k}{d} p_i^d (1-p_i)^{k-d} f(p_i) \mathrm{d}p_i} = \frac{d+1}{k+N} =$$

$$\frac{d}{k}\left(\frac{k}{k+N}\right) + \frac{1}{k}\left(\frac{k}{k+N}\right).$$

This suggests a type of production process in which the defects are assigned to the items sequentially and a given defect's chance of being assigned to an item increases as the fraction of defects that were already assigned to that item, d/k, increases. In other words, the presence of a large fraction of defects in an item will "atract" the next defect, increasing the chance that it will be assigned to that item. This "positive correlation" among defects is realistic in many production scenarios.

Sampling distribution

We have seen that, when applying the principle of indifference in the space of $\binom{N+S-1}{S}$ different configurations, the analyst obtains:

$$p(y_1, \ldots, y_N \mid N, S) = \frac{1}{\binom{N+S-1}{S}} \boldsymbol{I}\left(\sum_{i=1}^{N} y_i = S\right).$$

Suppose that a sample of size $n = 1$ item is selected. Let $p(y \mid n = 1, N, S)$ be the probability that the chosen item has y defects. Since y defects must be placed in one item and the remaining $S - y$ defects can be placed in the remaining $(N-1)$ items in $\binom{N-1+S-y-1}{S-y}$ different ways, each of them with probability $\binom{N+S-1}{S}^{-1}$, it follows that:

$$p(y \mid n=1, N, S) = \frac{\binom{N+S-y-2}{S-y}}{\binom{N+S-1}{S}} .$$

This is the "*finite version*" of the geometric distribution.

If the population of items and defects increases without bound, that is, if $N \to \infty$ and $S \to \infty$ so that $\frac{S}{N} = \theta$ is bounded, we may write:

$$p(y \mid n=1, N, S) = \frac{(N+S-y-2)!}{(S-y)!(N-2)!} \frac{S!(N-1)!}{(N+S-1)!} \to \frac{\theta^y}{(1+\theta)^{y+1}} ,$$

as $N \to \infty$, $S \to \infty$ and $\frac{S}{N} = \theta$. Here θ is the average number of defects per item.

In other words, when the population of items and defects increases without bound so that $\frac{S}{N} = \theta$, the probability distribution of the number of defects on the chosen item, y is given by a geometric distribution with parameter $(1+\theta)^{-1}$:

$$p(y \mid n=1, N, S) = \frac{\theta^y}{(1+\theta)^{y+1}} , \quad y = 0, 1, 2, \ldots$$

Now, consider a sample of size n. Let $p(y_1, y_2, \ldots, y_n \mid N, S, n)$ be the probability that the first item has y_1 defects, the second item has y_2 defects, ..., and the n-th item has y_n defects, and let $s = \sum_{i=1}^{n} y_i$ be the number of defects in the sample.

Since y_1 defects must be placed in the first item, ..., y_n defects must be placed in the n-th item, and the remaining $(S-s)$ defects can be placed in the remaining $(N-n)$ items in $\binom{N-n+S-s-1}{S-s}$ different ways, each of them with probability $\binom{N+S-1}{S}^{-1}$, it follows that:

$$p(y_1, y_2, \ldots, y_n \mid n, N, S) = \frac{\binom{N+S-n-s-1}{S-s}}{\binom{N+S-1}{S}} .$$

This means that all kinds of allocations of the s defects in the n items are equally likely. Here $p(y_1, y_2, \ldots, y_n \mid N, S)$ is Schur-constant (see Marshall and Olkin, 1979).

If we are interested in the distribution of s, the number of defects in the sample, we obtain:

$$p(s \mid n, N, S) = \binom{n+s-1}{s} \frac{\binom{N+S-n-s-1}{S-s}}{\binom{N+S-1}{S}}$$

since $\binom{n+s-1}{s}$ is the number of different vectors $(y_1, y_2, \ldots, y_n)$ such that $\sum_{i=1}^{n} y_i = s$. This is the "*finite version*" of the negative binomial distribution. When $N \to \infty$ and $S \to \infty$ so that $\frac{S}{N} = \theta$ is bounded, then

$$\begin{aligned} p(s|n, N, S) &= \binom{n+s-1}{s} \frac{(N+S-n-s-1)!}{(S-s)!(N-n-1)!} \frac{S!(N-1)!}{(N+S-1)!} \\ &\to \binom{n+s-1}{s} \frac{\theta^s}{(1+\theta)^{s+n}} \end{aligned}$$

for $s = 0, 1, 2, \ldots$, which is the probability function of a negative binomial distribution with parameters n and $(1+\theta)^{-1}$.

5. THE BINOMIAL DISTRIBUTION

Assume a population of N items which may be classified as defective or non defective and, in addition, assume that S items in the population are defective ($S \leq N$). There are $\binom{N}{S}$ different arrangements of defectives in the population. If all arrangements are judged equally likely, then each will have probability $\binom{N}{S}^{-1}$.

If y_i ($y_i = 0$ or 1) is the number of defects in the i-th item, the probability that the items $1, 2, \ldots, N$ will have $y_1, y_2, \ldots, y_N$ defects (where $\sum_{i=1}^{N} y_i = S$) is:

$$p(y_1, \ldots, y_N \mid N, S) = \frac{1}{\binom{N}{S}} I\left(\sum_{i=1}^{N} y_i = S\right).$$

This expression indicates that all kinds of allocations of defects in items are equally likely, that is, $p(y_1, \ldots, y_N \mid N, S)$ is Schur-constant (see Marshall and Olkin, 1979).

Sampling distribution

Now suppose we select a sample of size n. Let $p(x \mid n, S, N)$ be the probability that the sample has x defective items, where, obviously, $\max\{0, S+n-N\} \leq x \leq \min\{n, S\}$. Since x defective items can be selected to be in the sample in $\binom{n}{x}$ ways and $(S-x)$ defective items can remain among the $(N-n)$ items in $\binom{N-n}{S-x}$ ways, we may write:

$$p(x \mid n, S, N) = \frac{\binom{n}{x}\binom{N-n}{S-x}}{\binom{N}{S}}$$

for $\max\{0, S+n-N\} \leq x \leq \min\{n, S\}$, which is a hypergeometric distribution, the "*finite version*" of the binomial distribution.

If the population of defects and items increase without bound so that $N \to \infty$, $S \to \infty$ and $\frac{S}{N} = p$, $(p \leq 1)$, then

$$\begin{aligned} p(x|n, S, N) &= \binom{n}{x} \frac{(N-n)!}{(S-x)![N-n-(S-x)]!} \\ &\frac{S!(N-S)!}{N!} \to \binom{n}{x} p^x (1-p)^{n-x} = p(x|n, p)\ , \end{aligned}$$

which is the probability function of a binomial distribution with parameters n and p, where p is the chance that an item is defective.

6. FINAL REMARKS

This paper illustrates the natural way of assessing probabilistic models for a number of different production scenarios. Instead of choosing traditional probability distributions and looking for a situation that "fits" the distribution, a probability model is constructed based on physical characteristics underlying the production process. According to this approach, the parameters of interest, λ, θ, and p, are not abstract entities. On the contrary, they are defined operationally, that is, they are measurable functions of quantities that can be observed (N and S).

7. ACKNOWLEDGMENT

This research was done while the first author was visiting the department of Statistics of the University of São Paulo. It was supported by the National Science Foundation under Grant #DDM-9209344 and by FAPESP: Fundação de Amparo à Pesquisa do Estado de São Paulo, under Grant #91/3527-9.

REFERENCES

Barlow, R. E. and Mendel, M. B. (1992). De Finetti-type representations for life distributions. *J. Amer. Statist. Assoc.* **87**, 1116–1122.

Feller, W. (1957). *An Introduction to Probability Theory and its Applications* **I**, 2.ed. New York: Wiley

Hayakawa, Y. (1994). The construction of new bivariate exponential distributions from a Bayesian perspective. *J. Amer. Statist. Assoc.* **89**, 1044–1049.

Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. San Diego: Academic Press.

Mendel, M. B. (1989). *Development of Bayesian Parametric Theory with Applications to Control.*

Montgomery, D. C. (1991). *Introduction to Statistical Quality Control*. 2.ed. New York: Wiley