Reliability Estimation in Coherent Systems

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Preface

Usually, methods evaluating system reliability require engineers to quantify the reliability of each of the system components. For series and parallel systems, there are some options to handle the estimation of each component's reliability. We will treat the reliability estimation of complex problems of two classes of coherent systems: series-parallel, and parallel-series. In both of the cases, the component reliabilities may be unknown. We will present estimators for reliability functions at all levels of the system (component and system reliabilities). Nonparametric Bayesian estimators of all sub-distribution and distribution functions are derived, and a Dirichlet multivariate process as a prior distribution is presented. Parametric estimator of the component's reliability based on Weibull model is presented for any kind of system. Also, some ideas in systems with masked data are discussed.

This is a first version of the manuscript. We are sure that is necessary many improvements to this text. We are improving, and probably we will have a new version soon.

May 23, 2018.

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Chapter 1

Introduction

In engineering, the quality of the produced system is of great interest. In this sense, the reliability study has been object of research in last years. Among important works, Barlow e Proschan (1981) is highlight about theory of reliability. Barlow and Proschan's book shows concepts and theories about reliability, presenting important results when components and system reliabilities are known. In situations where component's reliability is unknown, statistical inference can be suitable considered to estimate system and components reliabilities and it will be discussed in the sequel of this book.

As a motivation, the reliability estimation of an automatic coffee machine is presented. The machine has two causes of failure: 1. the failure of the component that grinds the grain; or 2. the failure of the heating water component. Clearly, the failure of any component, 1 or 2, leads the coffee machine failure. The failure of a component implies that the possible future failure time of the other becomes invisible, i.e. a censored data. Statistical inference for the reliability of the machine depends on

both marginal components models. The reliability study of components allows for one-off actions on the components that need to be improved in order to maximize system performance, rather than changing the entire system, generating less costs, time and unnecessary effort. Hence, inferences for both components are needed.

Statistical inference of component reliability is not an easy task: censorship, dependence and unequal distributions are some of the troubles. Considering a sample of the coffee machine example for which all n sample units are observed up to death. Every sample unit will produce a component failure time and a censored failure time to the other component. Both components failing at the same time is considered unlikely in such situations. In this example, the sample will produce n failure times observations and n censored times for the two components in test. Relative to component failure time, it is reasonable to say that the two components are not identically distributed: probably one of the components may suffer more censors than the other. It is common that only one component is responsible for the system failure at time t, implying that all the remaining components are censored also at time t, although the types of censor could be different. In general, the number of censored observations should be higher than the uncensored ones.

The reliabilities of a system and its components also depend on the structure of the system, that is, the way that components are interconnected. The coffee machine is a series system of two components, a simple case known as competing risks problem. The illustration of a system structure can be considered by what is known as the block diagram, where each component of the system is illustrated by a block. Figure 1.1

is the block diagram of a series system with four components - at the time the system fails only one component is uncensored and the other three components are right-censored at the system failure time, that is, they still could continue to work after system fail.
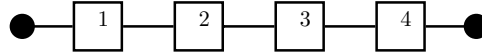
Figure 1.1: *Series system with four components.*

Suppose a system of m components and X_j denoting the failure time of the jth component, $j = 1, \ldots, m$. Let T the random variable that represents the system failure time. The function that defines T in relation to $X_1, \ldots, X_m$ depends on the system structure. For a system with the structure represented in Figure 1.1, for instance, $m = 4$ and $T = \min\{X_1, X_2, X_3, X_4\}$.

Consider initially that a random sample of n systems with the structure in Figure 1.1 is observed and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ being a sample of the random variable T. The goal is to estimate the reliabilities of components involved in this series structure. At the system failure, however, not all components would have their failure time observed. In addition, a particular component may be responsible for system failures in some sample units and not in the remaining ones, cases of right-censored observations.

When a system fails, the failure time of a given component j may not be observed, but its censored time of failure is. For all sample units, the system failure times $t_1, \ldots$ and t_n are recorded. Associated to each sample unit, let δ_i be the indicator of the component whose failure produced the system to fail, with $i = 1, \ldots, n$. At the time a series system fails, a given component can only be uncensored (responsible for system failure), that

is $\delta_i = j$, or right-censored at the system failure time, that is $\delta_i \neq j$, for $j = 1, \ldots, m$.

The data of $n = 10$ observed systems are presented in Table 1.1. For instance, system ID=1 failed at time 1.92 and component 1 is the first to fail, that is, $\delta_1 = 1$ and the others components are right-censored at 1.92.

Table 1.1: *Observed data of $n = 10$ series system with $m = 4$ components.*

System ID	t	δ
1	1.92	1
2	1.85	2
3	2.00	4
4	1.74	3
5	1.41	1
6	1.97	2
7	1.65	3
8	2.08	1
9	1.74	4
10	2.40	2

Considering a parametric model, let $R(x|\boldsymbol{\theta_j}) = \mathrm{P}(X_j > x|\boldsymbol{\theta_j})$ and $f(x|\boldsymbol{\theta_j})$ the reliability and density functions, respectively, and $\boldsymbol{\theta_j}$ is the parameter that can be either a scalar or a vector. For jth component, the likelihood function can be writen as

$$\mathrm{L}(\boldsymbol{\theta_j} \mid \mathbf{t}, \boldsymbol{\delta}) = \prod_{i=1}^{n} \left[f(t_i|\boldsymbol{\theta_j}) \right]^{\mathrm{I}_{\{\delta_i=j\}}} \left[R(t_i|\boldsymbol{\theta_j}) \right]^{1-\mathrm{I}_{\{\delta_i=j\}}}, \qquad (1.1)$$

where $\mathrm{I}_{\{TRUE\}} = 1$ or $\mathrm{I}_{\{FALSE\}} = 0$, $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ and $j = 1, \ldots, m$.

A parallel system as in Figure 1.2 works whenever at least one component is working. Again, only one component has its failure time uncensored, the other components are left-censored observations, that is, they had failed before system failure. For a system with the structure

represented in Figure 1.2, $m = 3$ and $T = \max\{X_1, X_2, X_3\}$.
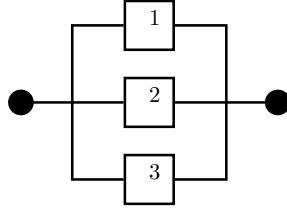
Figure 1.2: *Parallel system with three components.*

Consider that a random sample of $n = 7$ systems with the structure in 1.2 is observed and $\mathbf{t} = (t_1, t_2, \ldots, t_7)$ being a sample of the random variable T. Associated to each sample unit, let δ_i be the indicator of the component whose failure produced the ith system to fail, for $j = 1, 2, 3$ and $i = 1, \ldots, 7$. At the time a parallel system fails, a given component can only be uncensored (the last component to fail), that is $\delta_i = j$, or left-censored at the system failure time, that is, $\delta_i \neq j$, for $j = 1, \ldots, m$.

The parallel systems data are presented in Table 1.2. For instance, system ID=3 failed at time 4.93 and component 2 is the last to fail, that is, $\delta_3 = 2$ and the others components are left-censored at 4.93.

Table 1.2: *Observed data of $n = 7$ parallel system with $m = 3$ components.*

System ID	t	δ
1	0.18	1
2	1.15	3
3	4.93	2
4	0.01	2
5	1.01	1
6	1.51	3
7	1.74	2

For jth component and considering a parametric model, the likelihood

function can be writen as

$$\mathrm{L}(\boldsymbol{\theta_j} \mid \mathbf{t}, \boldsymbol{\delta}) = \prod_{i=1}^{n} \left[f(t_i|\boldsymbol{\theta_j}) \right]^{\mathrm{I}_{\{\delta_i=j\}}} \left[1 - R(t_i|\boldsymbol{\theta_j}) \right]^{1-\mathrm{I}_{\{\delta_i=j\}}}, \qquad (1.2)$$

where $\mathrm{I}_{\{TRUE\}} = 1$ or $\mathrm{I}_{\{FALSE\}} = 0$, $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ and $j = 1, \ldots, m$.

The literature on reliability of either parallel or series systems is abundant; different solutions have been presented. Salinas-Torres *et al.* (1997), Salinas-Torres *et al.* (2002), Polpo e Pereira (2009) and Polpo e Sinha (2011) discussed the Bayesian nonparametric statistics for series and parallel systems. Under Weibull probability distributions, Bayesian inferences for system and component reliabilities were introduced by Polpo *et al.* (2009) and Bhering *et al.* (2014) presented a hierarchical Bayesian Weibull model for components' reliability estimation in series and parallel systems, proposing an useful computational approach. Using simulation for series systems, Rodrigues *et al.* (2012), considering Weibull families, compared three estimation types: Kaplan-Meier, Maximum Likelihood and Bayesian Plug-in Estimators. Polpo *et al.* (2012) performed a comparative study about Bayesian estimation of a survival curve.

Series and parallel systems are particular cases of a class of system called coherent. A system is said to be coherent if all components are relevant, that is, all components play any role in the functional capacity of the system, and the structure function is nondecreasing in each component, that is, the system will not become worse than before if a failed component is replaced by another that works.

An important property is presented by Barlow e Proschan (1981) that

every coherent system can be written as series-parallel system (SPS) representation and as parallel-series system (PSS) representation. Consider the PSS in Figure 1.3a and Figure 1.3b presents its SPS representation. The SPS in Figure 1.4a has its PSS representation in Figure 1.4b.

(a)   **(b)**
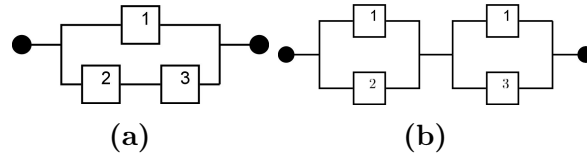
Figure 1.3: *(a) PSS (b) SPS representation of system in (a).*

(a)   **(b)**
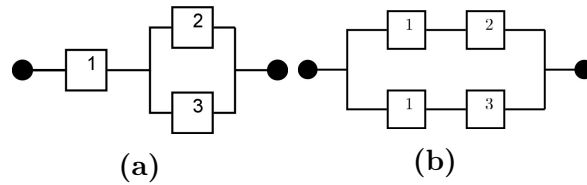
Figure 1.4: *(a) SPS (b) PSS representation of system in (a).*

Considering this celebrated property, Polpo *et al.* (2013) introduced Bayesian nonparametric statistics for a class of coherent system in order to estimate components reliabilities. They restricted themselves to cases for which no component appears more than twice in parallel-series and series-parallel representations, under assumption that two components or more can not fail at the same instant of time. However, it is common that, in the representation of the system, some components appear in two different places within it. For instance, consider again the representation in Figure 1.3b (or Figure 1.4b). We have the reliabilities of four components to estimate. However, two of them are in fact the same component (component 1), and they will fail at the same time, which violates the assumption. For this reason, it is important to have the estimators for both SPS and PSS that give a wide variety of represen-

tations. If one of these representations does not violate the assumptions, then the proposed Bayesian nonparametric estimator can be used. This nonparametric approach is presented with details in Chapter 2.

Figure 1.5 is the bridge system described in the literature (Barlow e Proschan, 1981) and Figure 1.6 illustrates its SPS and PSS representations. Note that each of the five components appears twice for both representations. Another interesting structure is the k-out-of-m system - work only if at least k out of the m components work. For instance, Figure 1.7 considers the simple 2-out-of-3 case into SPS and PSS representations. Note that each of the three components also appears twice in both combinations. Situations like these violate Polpo *et al.* (2013) assumption and their approach is not suitable anymore both for the PSS representation and for SPS representation. Thus, a solution to estimate the reliability of components in systems such as Figures 1.5 and 1.7 is to consider the parametric approach and the general likelihood function is developed in the sequel.
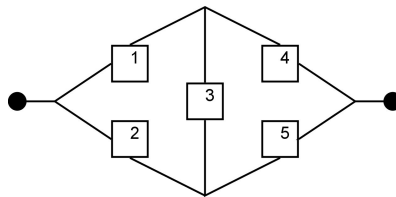
Figure 1.5: *Bridge structure.*

In likelihood functions (1.1) and (1.2) the jth component is susceptible only to right-censored or only to left-censored data, respectively. For a more general case, a component can be susceptible to both side of censoring. For instance, system 2-out-of-3 (Figure 1.7) - system work if at least 2 out of 3 components work. Consider that a system 2-out-of-3 is observed
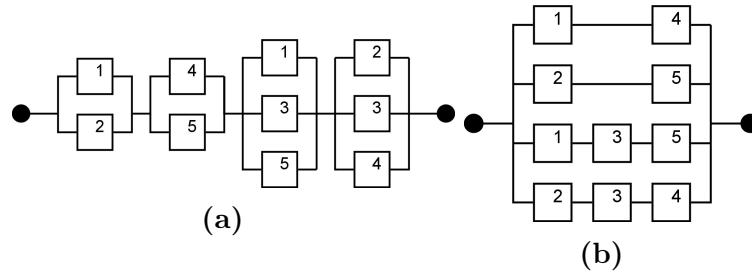
Figure 1.6: *(a) SPS bridge representation (b) PSS bridge representation.*
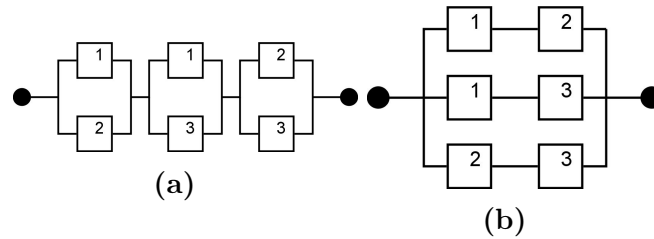
Figure 1.7: *(a) SPS 2-out-of-3 representation (b) PSS 2-out-of-3 representation.*

and component 1 failed first and component 3 failed in the sequence. At the moment of component 3 failure, the system failed. Thus, component 3 is uncensored, component 1 is left-censored data and component 2 is right-censored observation. Another 2-out-of-3 system is observed but for this system, component 2 was the first to fail (left-censored), component 1 was the last to fail (uncensored) and component 3 yet worked in system failure (right-censored). Note that each component is susceptible to be uncensored, left or right-censored failure time.

Another kind of censoring could also occur: suppose a machine failure time (a sample unit) is in an interval $(l, u) - l$ for the observed lower limit and u for the upper limit. If two or more components failed, they are all interval censored in (l, u).

To generalize the notation for all cases of component failure and censoring, consider the following notation: for a specific component j of the

system unit i, let (l_{ji}, u_{ji}) be a general interval of time, in which

- $l_{ji} = u_{ji} = t_i$, if the j-th component failure time causes the i-th system failure time;

- $l_{ji} = t_i$ and $u_{ji} = \infty$, if the j-th component is right-censored at t_i;

- $l_{ji} = 0$ and $u_{ji} = t_i$, if the j-th component is left-censored at t_i;

- $0 < l_{ji} < u_{ji} < \infty$, if the j-th component is interval-censored.

Consider that a random sample of $n = 10$ systems with the structure in Figure 1.7 is observed. The data are presented in Table 1.3. For instance, system ID=2 failed at time 2.09 and the failure of component 2 causes the system failure ($l_{22} = 2.09$ and $u_{22} = 2.09$), component 1 had failed before time 2.09 ($l_{12} = 0$ and $u_{12} = 2.09$) and component 3 is right-censored at time 2.09 ($l_{32} = 2.09$ and $u_{32} = \infty$).

Table 1.3: *Observed data of $n = 10$ 2-out-of-3 systems.*

System ID	Component 1		Component 2		Component 3	
	l	u	l	u	l	u
1	1.95	1.95	1.95	∞	0	1.95
2	0	2.09	2.09	2.09	2.09	∞
3	3.56	∞	3.56	3.56	0	3.56
4	2.55	∞	0	2.55	2.55	2.55
5	1.89	1.89	1.89	∞	0	1.89
6	3.01	∞	0	3.01	3.01	3.01
7	2.43	2.43	0	2.43	2.43	∞
8	0	1.51	1.51	1.51	1.51	∞
9	3.55	3.55	3.55	∞	0	3.55
10	2.35	∞	0	2.35	2.35	2.35

To complete the theoretical environment, let $f(\cdot|\boldsymbol{\theta_j})$ and $R(\cdot|\boldsymbol{\theta_j})$ the density and reliability functions, respectively, and $\boldsymbol{\theta_j}$ is the parameter

that can be either a scalar or a vector. Using the above notation, the likelihood function is as follow:

$$L(\boldsymbol{\theta_j} \mid \mathbf{l_j}, \mathbf{u_j}) = \prod_{i=1}^{n} \left[f(l_{ji}|\boldsymbol{\theta_j}) \right]^{I_{\{l_{ji}=u_{ji}\}}} \left[R(l_{ji}|\boldsymbol{\theta_j}) - R(u_{ji}|\boldsymbol{\theta_j}) \right]^{1-I_{\{l_{ji}=u_{ji}\}}} (1.3)$$

where $I_{\{TRUE\}} = 1$ or $I_{\{FALSE\}} = 0$; $\mathbf{l_j} = (l_{j1}, \ldots, l_{jn})$ and $\mathbf{u_j} = (u_{j1}, \ldots, u_{jn})$.

A parametric approach which considers the likelihood function (1.3) for estimation of components' reliabilities involved in any kind of coherent system, from the simplest to the most complex structures, is presented in Chapter 3. The available information are the failure time of system and the status of each component at system failure instant. This approach does not need the supposition of identically distributed components lifetimes. The main assumption is that components' lifetimes are mutually independent and the components lifetime distributions are the three-parameter Weibull, a very general distribution that can approximate most of the lifetimes distributions. The paradigm is the Bayesian one. Another advantage of the Weibull is that, in our paradigm, even with improper priors, the posterior distributions turn out to be proper. The presented mechanism of calculus can well be used for any other family of distributions whenever proper priors are used.

Chapters 2 and 3 will address the problem of component estimation in coherent systems under the Bayesian paradigm in nonparametric and parametric approaches, respectively. In both chapter the status of each component at the time of system failure is considered to be known. However, identifying which component fail caused the failure of a given system

can be a difficult task. In special situations, we can only establish that failed components belong to small sets of components. Cases like this are known as masked data failure cause and it is usually due to limited resources for the diagnosis of the cause of the failure. As an example of masked data problem, Basu *et al.* (1999) cited situations of failures of large computer systems, where the analysis is often performed in such a way that a small subset of components is identified as the cause of failure. In an attempt to repair the system as quickly as possible, the entire subset of components is replaced and the component responsible for the failure can not be investigated.

In Chapter 4 the masked data problem formulation is developed and a Bayesian three-parameter Weibull model for components' reliabilities in masked data scenario is presented. This model is general and it can be considered for components involved in any coherent system.

Finally, the datasets considered in Chapters 2, 3 and 4 are presented in Appendixs A, B and C, respectively.

Chapter 2

Nonparametric

A nonparametric estimator for all the reliability functions involved in the series-parallel system (SPS) and parallel-series system (PSS) under the assumptions that the components reliabilities are unknown is presented; the only available information are the failure times of the system and the component that produced the failure. The required assumptions are mutually independent components failure times and that two or more components cannot fail at same instant of time.

In Section 2.1 are presented the probability results necessary for the development of the estimator. Section 2.2 is devoted to the construction of the nonparametric Bayesian estimator for SPS and PSS with three components ($m = 3$). In Section 2.3 the results are extended to a more general case of $m \geq 4$; and in Section 2.4 the estimator is used in simulated datasets and illustrated its qualities.

2.1   Probability relations

In this section, important results and properties of the PSS and SPS are presented. Before it, we present these results for series and parallel systems that will facilitate the understanding of the results for PSS and SPS, once series and parallel systems are simpler.

2.1.1   Series and parallel systems

First consider a parallel system with m components. Let X_j be the failure time of jth component with marginal distribution function (DF) F_j and $T = \max\{X_1, \ldots, X_m\}$ be the system failure time. The indicator of the component whose failure produced the system to fail is $\delta = j$ when $T = X_j$, $j = 1, \ldots, m$. The jth sub-distribution function evaluated at a time t is the probability that the system survives at most to time t and the last component to fail is the jth component, that is, $F_j^*(t) = \mathrm{P}(\mathrm{T} \leq \mathrm{t}, \delta = \mathrm{j})$.

Let $F(t_1, \ldots, t_m) = P(X_1 \leq t_1, \ldots, X_m \leq t_m)$ be the joint distribution function, in which continuous partial derivatives are assumed over all arguments. The following theorem establishes the relation between the joint distribution function with the jth sub-distribution $F_j^*(t)$.

Theorem 1 *The derivative of $F_j^*(t)$, $dF_j^*(t)/dt$, is equal to the partial derivative of $F(t_1, \ldots, t_m)$ at the jth component, evaluated at $t_1 = t_2 = \ldots = t_m = t$.*

Because the life of the components are assumed to be mutually s-

independent,

$$F(t_1, \ldots, t_m) = \prod_{j=1}^{m} F_j(t_j). \tag{2.1}$$

Using the fact in (2.1) and the Theorem 1,

$$\frac{d}{dt} F_j^*(t) = u_j(t) \prod_{j=1}^{m} F_j(t), \tag{2.2}$$

where u_j is the reversed hazard rate (RHR) of the jth component:

$$u_j(t) = \frac{f_j(t)}{F_j(t)} = \frac{d}{dt} \ln F_j(t). \tag{2.3}$$

From (2.3) one can write

$$F_j(t) = \exp\left\{ -\int_t^\infty u_j(y) dy \right\}. \tag{2.4}$$

Letting $u(y) = \sum_{j=1}^{m} u_j(y)$, (2.2) becomes

$$\frac{d}{dt} F_j^*(t) = u_j(t) \exp\left\{ -\int_t^\infty u(y) dy \right\}, \tag{2.5}$$

Taking now the sum for $j = 1, \ldots, m$ in both sides of (2.5), we obtain

$$\begin{aligned} \sum_{j=1}^{m} \frac{d}{dt} F_j^*(t) &= u(t) \exp\left\{ -\int_t^\infty u(y) dy \right\} \\ &= \frac{d}{dt} \exp\left\{ -\int_t^\infty u(y) dy \right\}. \end{aligned} \tag{2.6}$$

Consequently,

$$\sum_{j=1}^{m} F_j^*(t) = \exp\left\{ -\int_t^{\infty} u(y)dy \right\},$$

which combined with (2.5) leads to

$$u_j(t) = \frac{dF_j^*(t)/dt}{\sum_{j=1}^{m} F_j^*(t)}. \tag{2.7}$$

Finally, (2.4) implies

$$F_j(t) = \exp\left\{ -\int_t^{\infty} \frac{dF_j^*(y)}{\sum_{j=1}^{m} F_j^*(y)} \right\}, \tag{2.8}$$

that is, the relationship of interest between marginal distribution functions and sub-distribution functions.

Unfortunately, the expression in (2.8) does not work for the case with jump points. To obtain a version of (2.8) in the presence of jumps, we introduce the following definition and theorem.

Definition 1 *For simplicity, consider the case of $m = 2$. The function $\Phi_p\left(F_1^*, F_2^*, t\right)$ based on the sub-distributions F_1^* and F_2^* is*

$$\Phi_{p2}\left(F_1^*, F_2^*, t\right) \equiv \exp\left\{ \oint_t^{\infty} \frac{-dF_1^*(v)}{\sum_{j=1}^{2} F_j^*(v)} \right\} \prod_{v>t}^{F_1^*} \left[\frac{\sum_{j=1}^{2} F_j^*(v^-)}{\sum_{j=1}^{2} F_j^*(v^+)} \right],$$

where $\oint g(s)ds$ is integration over disjoint open intervals that do not include the jump points of $g(\cdot)$ and $\prod^{G}$ is product over jump points of $G(\cdot)$.

The next result, although restricted to $m = 2$, extends expression (2.8) in the sense that it can include disjoint jump points.

Theorem 2 *The sub-distribution functions F_1^* and F_2^* determine (uniquely) the distribution function F_1 for $t \leq t^*$ by $F_1(t) = \Phi_{p2}(F_1^*, F_2^*, t)$.*

An analogous development can be performed for a series system with m components, in which $T = \min\{X_1, \ldots, X_m\}$ and $\delta = j$, if $T = X_j$. The version of (2.8) for a series system is given by (Salinas-Torres *et al.*, 2002):

$$F_j(t) = 1 - \exp\left\{ \int_0^t \frac{dR_j^*(y)}{\sum_{j=1}^m R_j^*(y)} \right\}, \tag{2.9}$$

in which $R_j^*(t) = \mathrm{P}(\mathrm{T} > \mathrm{t}, \delta = \mathrm{j})$ is the sub-reliability function for jth component.

Unfortunately, the expression in (2.9) does not work for the case with jump points. To obtain a version of (2.9) in the presence of jumps, we introduce the following definition and theorem.

Definition 2 *For simplicity, consider the case of $m = 2$. The function $\Phi_{s2}(R_1^*, R_2^*, t)$ based on the sub-reliability functions R_1^* and R_2^* is*

$$\Phi_{s2}(R_1^*, R_2^*, t) \equiv 1 - \exp\left\{ \oint_0^t \frac{\mathrm{d}R_1^*(v)}{\sum\limits_{j=1}^2 R_j^*(v)} \right\} \prod_{v>t}^{F_1^*} \left[\frac{\sum\limits_{j=1}^2 R_j^*(v^+)}{\sum\limits_{j=1}^2 R_j^*(v^-)} \right],$$

where $\oint g(s)\mathrm{d}s$ is integration over disjoint open intervals that do not include the jump points of $g(\cdot)$ and $\prod^G$ is product over jump points of $G(\cdot)$.

The next result, although restricted to $m = 2$, extends expression (2.8) in the sense that it can include disjoint jump points.

Theorem 3 *The sub-reliability functions R_1^* and R_2^* determine (uniquely) the distribution function F_1 for $t \leq t^*$ by $F_1(t) = \Phi_{s2}(R_1^*, R_2^*, t)$.*

For more details about relations among the distributions and sub-distribution functions (sub-reliability functions) can be found in Salinas-Torres *et al.* (2002) and Polpo e Sinha (2011) for series system and Polpo e Pereira (2009) for parallel system.

In next Sub-section the relations among the distributions and sub-distribution functions are presented for a more general class of systems - SPS and PSS.

2.1.2   PSS and SPS

We restrict ourselves to a system with three components ($m = 3$), given in Figure 1.3a and in Figure 1.4a.

Let X_1, X_2 and X_3 be the lifetimes of three components of an PSS and SPS with marginal distribution functions (DF) F_1, F_2, and F_3, respectively. The restriction here is that the three sets of jump points of F_1, F_2, and F_3 must be disjoint. The indicator of the component whose failure produced the system to fail is $\delta = 1$ when $T = X_1$, $\delta = 2$ when $T = X_2$, and $\delta = 3$ when $T = X_3$. Let $F_j^*(t) = \mathrm{P}(\mathrm{T} \leq \mathrm{t}, \delta = \mathrm{j})$ be the sub-distribution function of the jth component and $F(\cdot)$ the distribution function of the system. The following properties can be proved.

Property 1 *The sub-distribution functions (SDF) F_1^*, F_2^*, and F_3^* determine the DF of the system,*

$$F(t) = F_1^*(t) + F_2^*(t) + F_3^*(t). \qquad (2.10)$$

Property 2   *1. $F_1^*(+\infty) = \Pr(\delta = 1)$;*

2. $F_2^*(+\infty) = \Pr(\delta = 2)$;

3. $F_3^*(+\infty) = \Pr(\delta = 3)$;

4. $F_1^*(+\infty) + F_2^*(+\infty) + F_3^*(+\infty) = 1$.

Property 3 *The set of jump points F_j^* and F_j are the same, where $j = 1, 2, 3$. Because F_1, F_2, and F_3 have disjoint set of jump points, so have F_1^*, F_2^*, and F_3^*.*

Property 4 *If $\min(F_1(t), F_2(t), F_3(t)) < 1$ for $t < t^*$, and 1 for $t \geq t^*$, then t^* is the largest support point of the system.*

The lifetime of the SPS is $T = \min(X_1, \max(X_2, X_3))$ and the system reliability of s-independent components is

$$R(t) = [1 - F_1(t)][1 - F_2(t)F_3(t)]. \qquad (2.11)$$

The lifetime of the PSS is $T = \max(X_1, \min(X_2, X_3))$ and the system reliability of s-independent components is

$$R(t) = 1 - F_1(t)\left\{1 - [1 - F_2(t)][1 - F_3(t)]\right\}. \qquad (2.12)$$

Property 5 *The SDF of the SPS can be expressed using the marginal DF of the components by*

$$F_1^*(t) = \int_0^t [1 - F_2(t)F_3(t)]\mathrm{d}F_1(t),$$

$$F_2^*(t) = \int_0^t [1 - F_1(t)]F_3(t)\mathrm{d}F_2(t),$$

$$F_3^*(t) = \int_0^t [1 - F_1(t)]F_2(t)\mathrm{d}F_3(t), \qquad (2.13)$$

and the SDF of the PSS can be expressed using the marginal DF of the components by

$$F_1^*(t) = \int_0^t \{1 - [1 - F_2(t)][1 - F_3(t)]\}\mathrm{d}F_1(t),$$

$$F_2^*(t) = \int_0^t F_1(t)[1 - F_3(t)]\mathrm{d}F_2(t),$$

$$F_3^*(t) = \int_0^t F_1(t)[1 - F_2(t)]\mathrm{d}F_3(t). \qquad (2.14)$$

Our interest is to obtain the inverse of (2.13) and (2.14); that is, to express the DFs F_1, F_2 and F_3 as a function of the SDF $(F_1^*,\ F_2^*,\ F_3^*)$. These inverses are presented with the following definitions and theorems.

Definition 3 *The functions $\Phi_s(F_1^*, F_2^*, F_3^*, t)$, and $\Phi_p(F_1^*, F_2^*, F_3^*, t)$ based*

on sub-distributions F_1^*, F_2^*, and F_3^* are

$$\Phi_s(F_1^*, F_2^*, F_3^*, t) \equiv 1 - \left\{ \exp\left[\oint_0^t \frac{-\mathrm{d}F_1^*(v)}{1 - \sum_{j=1}^3 F_j^*(v)}\right] \right.$$

$$\left. \prod_{v \leq t}^{F_1^*} \left[\frac{1 - \sum_{j=1}^3 F_j^*(v^+)}{1 - \sum_{j=1}^3 F_j^*(v^-)}\cdot\right] \right\},$$

$$\Phi_p(F_1^*, F_2^*, F_3^*, t) \equiv \exp\left\{\oint_t^\infty \frac{-\mathrm{d}F_1^*(v)}{\sum_{j=1}^3 F_j^*(v)}\right\} \prod_{v > t}^{F_1^*} \left[\frac{\sum_{j=1}^3 F_j^*(v^-)}{\sum_{j=1}^3 F_j^*(v^+)}\right].$$

The functions Φ_s (for a series system), and Φ_p (for a parallel system) are the versions with three components for those presented in Polpo e Sinha (2011) and Polpo e Pereira (2009), respectively. First, Theorem 4 states the relation between F_1 and F_1^*, F_2^*, and F_3^*.

Theorem 4 *The SDF F_1^*, F_2^*, and F_3^* determine (uniquely) the DF F_1 of an SPS for $t \leq t^*$ by $F_1(t) = \Phi_s(F_1^*, F_2^*, F_3^*, t)$, and the DF F_1 of a PSS for $t \leq t^*$ by $F_1(t) = \Phi_p(F_1^*, F_2^*, F_3^*, t)$.*

The next definition gives the functions Φ_{sp} (for the SPS), and Φ_{ps} (for the PSS).

Definition 4 *The functions $\Phi_{sp}(F_1^*, F_2^*, F_3^*, t)$, and $\Phi_{ps}(F_1^*, F_2^*, F_3^*, t)$, based on sub-distributions F_1^*, F_2^*, and F_3^*, are*

$$\Phi_{sp}(F_1^*, F_2^*, F_3^*, t) \equiv$$

$$\Phi_{ps}(F_1^*, F_2^*, F_3^*, t) \equiv$$
$$\exp\left\{ \oint_t^\infty \frac{-\mathrm{d}F_2^*(v)}{\sum_{j=1}^3 F_j^*(v) - \Phi_s(F_1^*, F_2^*, F_3^*, v)} \right\}$$
$$\prod_{v>t}^{F_2^*} \left[\frac{\sum_{j=1}^3 F_j^*(v^-) - \Phi_s(F_1^*, F_2^*, F_3^*, v^-)}{\sum_{j=1}^3 F_j^*(v^+) - \Phi_s(F_1^*, F_2^*, F_3^*, v^+)} \right],$$
$$1 - \left\{ \exp\left[\oint_0^t \frac{-\mathrm{d}F_2^*(v)}{\Phi_p(F_1^*, F_2^*, F_3^*, v) - \sum_{j=1}^3 F_j^*(v)} \right] \right.$$
$$\left. \prod_{v\le t}^{F_2^*} \left[\frac{\Phi_p(F_1^*, F_2^*, F_3^*, v^+) - \sum_{j=1}^3 F_j^*(v^+)}{\Phi_p(F_1^*, F_2^*, F_3^*, v^-) - \sum_{j=1}^3 F_j^*(v^-)} \right] \right\}.$$

Theorem 5 *The SDF F_1^*, F_2^*, and F_3^* determine (uniquely) the DF F_2 of an SPS for $t \le t^*$ by $F_2(t) = \Phi_{sp}(F_1^*, F_2^*, F_3^*, t)$, and the DF F_2 of a PSS for $t \le t^*$ by $F_2(t) = \Phi_{ps}(F_1^*, F_2^*, F_3^*, t)$.*

Note that Theorem 5 can be easily rewritten to obtain the relation of DF F_3 and the SDF.

The proofs of Theorems 4 and 5 can be seen in Polpo *et al.* (2013).

Theorem 5 provides an important relation between the SDF and DF, for both SPS and PSS. Using this result, in the next section, we have developed the nonparametric Bayesian estimator for the DF of the system's components.

2.2   Bayesian Analysis

This section describes a Bayesian reliability approach to SPS and PSS. We have derived a nonparametric Bayesian estimator of the distribution function using the multivariate Dirichlet process (Salinas-Torres $et\ al.$, 2002). From Property 1, we have that the sub-distribution functions are related to the system distribution function by a sum. Considering that $F_1^*(t) + F_2^*(t) + F_3^*(t) + (1 - F(t)) = 1$, we have the restriction that these four quantities have a sum equal to 1, and that the set of possible points for $\{F_1^*(t), F_2^*(t), F_3^*(t), 1 - F(t)\}$ is the four-dimensional simplex, or $\{F_1^*(t), F_2^*(t), F_3^*(t)\}$ for the non-singular form. In this case, for a fixed t, we have that a natural prior choice is the Dirichlet distribution, and for any t, we have the Dirichlet multivariate process. In this Section a nonparametric estimator for the distribution function of the components in an SPS or in a PSS, and using the Dirichlet process, we have a complete distribution for the set $\{F_1^*(t), F_2^*(t), F_3^*(t)\}$. In this case, our parameters are the functions that we want to estimate, giving us a nonparametric framework.

Consider a sample of size n and the observed data are $(T_1, \delta_1), \ldots,$ (T_n, δ_n), in which $T_i = \min(X_{1i}, \max(X_{2i}, X_{3i}))$ for SPS and $T_i = \max(X_{1i}, \min(X_{2i}, X_{3i}))$ for PSS. Besides, $\delta_i = j$ if $T_i = X_{ji}$, for $i = 1, \ldots, n$ and $j = 1, 2, 3$. Equivalently, for each t, the random variables are observed:

$$nF_{jn}^*(t) = \sum_{i=1}^{n} \mathrm{I}(T_i \le t, \delta_i = j), \quad \text{for } j = 1, 2, 3,$$

in which $\mathrm{I}(A)$ is a indicator function of set A.

The function F_{jn}^* is empirical sub-distribution function of jth component. If $F_n(\cdot)$ is the empirical distribution function corresponding to the observations $T_1, \ldots, T_n$, thus for each t,

$$nF_n(t) = nF_{1n}^*(t) + nF_{2n}^*(t) + nF_{3n}^*(t).$$

For each t, let $k_j(t)$ the realization of $nF_{jn}^*(t)$, in which

$$k_j(t) = \sum_{i=1}^{n} \mathrm{I}(t_i \leq t, \delta_i = j), \quad \text{for } j = 1, 2, 3.$$

In this context, for each t, the likelihood function corresponds to the likelihood of a multinomial model $M(n; p_1(t), p_2(t), p_3(t), p_4(t))$ being $p_j(t) = F_j^*(t)$, for $j = 1, 2, 3$, and $p_4(t) = 1 - F(t) = 1 - \sum_{j=1}^{3} F_j^*(t)$, that is,

$$
\begin{aligned}
L &= \mathrm{P}(nF_{1n}^* = k_1(t), nF_{2n}^* = k_2(t), nF_{3n}^* = k_3(t)) \\
&\propto [F_1^*(t)]^{k_1(t)}[F_2^*(t)]^{k_2(t)}[F_3^*(t)]^{k_3(t)}[1 - F(t)]^{n - \sum_{j=1}^{3} k_j(t)} \quad (2.15)
\end{aligned}
$$

The prior distribution for $(F_1^*(\cdot), F_2^*(\cdot), F_3^*(\cdot))$ is constructed from the characterization of the multivariate Dirichlet process, defined in Salinas-Torres *et al.* (1997) and it may have the following simplified version.

Definition 5 *Let Ω be a sample space, $\alpha_1, \ldots, \alpha_m$ be finite positive measures defined over Ω, and $\rho = (\rho_1, \ldots, \rho_m)$ be a random vector having a Dirichlet distribution with parameters $(\alpha_1(\Omega), \ldots, \alpha_m(\Omega))$. Consider m Dirichlet processes, $P_1, \ldots, P_m$, with $P_j \sim \mathcal{D}(\alpha_j)$, $j = 1, \ldots, m$. All these*

processes and ρ are mutually s-independent random quantities. Define
$P^* = (P_1^*, \ldots, P_m^*) = (\rho_1 P_1, \ldots, \rho_m P_m)$. *The P^* is a Dirichlet multivariate process with parameter measures $\alpha_1, \ldots, \alpha_m$.*

In the context of SPS and PSS, consider $\Omega = (0, \infty)$, $\rho_j = \mathrm{P}(\delta = j)$ and $P_j(t) = \mathrm{P}(T \leq t | \delta = j)$, for $j = 1, 2, 3$. Then, the vector of components sub-distribution functions is $F^* = (F_1^*, F_2^*, F_3^*) = (\rho_1 P_1, \rho_2 P_2, \rho_3 P_3)$ and the prior distribution is given by

$$F^*(t) \sim \mathcal{D}(\alpha_1(0, t], \alpha_2(0, t], \alpha_3(0, t]; \sum_{j=1}^{3} \alpha_j(t, \infty)). \qquad (2.16)$$

Combining the prior distribution (2.16) and the likelihood function in (2.15), the posterior distribution of $F^*(t) = (F_1^*(t), F_2^*(t), F_3^*(t))$ is, for each t,

$$(F_1^*(t), F_2^*(t), F_3^*(t)) | Data \sim \mathcal{D}(\alpha_1(0, t] + nF_{1n}^*(t), \alpha_2(0, t] + nF_{2n}^*(t),$$
$$\alpha_3(0, t] + nF_{3n}^*(t); \sum_{j=1}^{3} \alpha_j(t, \infty) + n - \sum_{j=1}^{3} nF_{jn}^*(t)).$$

Thus, the posterior means of $F_j^*(t)$ and $F(t)$ are given by

$$\widehat{F}_j^*(t) = p_\alpha \frac{\alpha_j(0, t]}{\sum\limits_{\ell=1}^{3} \alpha_\ell(0, \infty)} + (1 - p_\alpha) F_{jn}^*(t), \qquad (2.17)$$

where $p_\alpha = \left(\sum\limits_{j=1}^{3} \alpha_j(0, \infty) \right) \Big/ \left(n + \sum\limits_{j=1}^{3} \alpha_j(0, \infty) \right)$, and

$$\widehat{F}(t) = \sum_{j=1}^{3} \widehat{F}_j^*(t). \qquad (2.18)$$

These Bayesian estimators are strongly s-consistent. For instance, using the Glivenko Cantelli Theorem (Billingsley, 1985), it can be shown that $\widehat{F}_j^*$ converges to F_j^* uniformly with probability 1.

If $\alpha_j(0, \infty) < \infty$, the Bayesian estimator of $\rho_j = \mathrm{P}(\delta = j)$ is given by

$$\widehat{\rho}_j = \lim_{t \uparrow \infty} \widehat{F}_j^*(t) = \frac{\alpha_j(0, \infty)}{n + \sum\limits_{\ell=1}^{3} \alpha_\ell(0, \infty)} + \frac{\sum\limits_{i=1}^{n} \mathrm{I}(\delta_i = j)}{n + \sum\limits_{\ell=1}^{3} \alpha_\ell(0, \infty)}. \tag{2.19}$$

Let the v $(\leq n)$ distinct order statistics of T be $T_{(1)}^\bullet < \ldots < T_{(v)}^\bullet$. Set $N_i = \sum\limits_{\ell=1}^{n} \mathrm{I}(T_\ell < T_{(i)}^\bullet)$, and $d_{ji} = \sum\limits_{\ell=1}^{n} \mathrm{I}(T_\ell = T_{(i)}^\bullet, \delta_\ell = j)$, $i = 1, \ldots, v$. Define

$$I_s(t) = \exp\left\{ \frac{1}{\sum\limits_{j=1}^{3} \alpha_j(0, \infty) + n} \int\limits_0^t \frac{-\mathrm{d}\alpha_1(0, s]}{1 - \widehat{F}(s)} \right\}, \tag{2.20}$$

$$\Pi_s(t) = \prod_{i: T_{(i)}^\bullet \leq t} \frac{\sum\limits_{j=1}^{3} \alpha_j(T_{(i)}^\bullet, \infty) + n - N_i - d_{1i}}{\sum\limits_{j=1}^{3} \alpha_j(T_{(i)}^\bullet, \infty) + n - N_i}. \tag{2.21}$$

$$I_{sp}(t) = \exp\left\{ \frac{1}{\sum\limits_{j=1}^{3} \alpha_j(0, \infty) + n} \int\limits_t^\infty \frac{-\mathrm{d}\alpha_2(0, s]}{\widehat{F}(s) - \widehat{F}_1(s)} \right\}, \tag{2.22}$$

$$\Pi_{sp}(t) = \prod_{i: T_{(i)}^\bullet > t} \frac{\frac{\sum\limits_{j=1}^{3} \alpha_j(0, T_{(i)}^\bullet] + N_i}{n + \sum\limits_{j=1}^{3} \alpha_j(0, \infty)} - \widehat{F}_1(T_{(i)}^\bullet)}{\frac{\sum\limits_{j=1}^{3} \alpha_j(0, T_{(i)}^\bullet] + N_i + d_{2i}}{n + \sum\limits_{j=1}^{3} \alpha_j(0, \infty)} - \widehat{F}_1(T_{(i)}^\bullet)}. \tag{2.23}$$

$$I_p(t) = \exp\left\{ \frac{1}{\sum\limits_{j=1}^{3} \alpha_j(0,\infty) + n} \int\limits_{t}^{\infty} \frac{-\mathrm{d}\alpha_1(0,s]}{\widehat{F}(s)} \right\}, \qquad (2.24)$$

$$\Pi_p(t) = \prod_{i:T_{(i)}^{\bullet}>t} \frac{\sum\limits_{j=1}^{3} \alpha_j(0,T_{(i)}^{\bullet}] + N_i}{\sum\limits_{j=1}^{3} \alpha_j(0,T_{(i)}^{\bullet}] + N_i + d_{1i}}. \qquad (2.25)$$

$$I_{ps}(t) = \exp\left\{ \frac{1}{\sum\limits_{j=1}^{3} \alpha_j(0,\infty) + n} \int\limits_{0}^{t} \frac{-\mathrm{d}\alpha_2(0,s]}{\widehat{F}_1(s) - \widehat{F}(s)} \right\}, \qquad (2.26)$$

and

$$\Pi_{ps}(t) = \prod_{i:T_{(i)}^{\bullet}\leq t} \frac{\widehat{F}_1(T_{(i)}^{\bullet}) - \frac{\sum\limits_{j=1}^{3} \alpha_j(0,T_{(i)}^{\bullet}] + N_i + d_{2i}}{n + \sum\limits_{j=1}^{3} \alpha_j(0,\infty)}}{\widehat{F}_1(T_{(i)}^{\bullet}) - \frac{\sum\limits_{j=1}^{3} \alpha_j(0,T_{(i)}^{\bullet}] + N_i}{n + \sum\limits_{j=1}^{3} \alpha_j(0,\infty)}}. \qquad (2.27)$$

The main result of this study is given in the following paragraph.

Theorem 6 *Suppose that $\alpha_1(0,\cdot), \alpha_2(0,\cdot), \alpha_3(0,\cdot)$ are continuous on (t,∞), for each $t > 0$, and F_1, F_2, and F_3 have no common discontinuities. Then, for $t \leq T_{(n)}$, and SPS, we have that*

$$\widehat{F}_1(t) = \mathrm{E}[F_1(t)|data] \qquad (2.28)$$

$$= \Phi_s(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t) = 1 - I_s(t)\Pi_s(t),$$

$$\widehat{F}_2(t) = \mathrm{E}[F_2(t)|data] \qquad (2.29)$$

$$= \Phi_{sp}(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t) = I_{sp}(t)\Pi_{sp}(t);$$

and, for PSS,

$$\widehat{F}_1(t) = \mathrm{E}[F_1(t)|data] \tag{2.30}$$
$$= \Phi_p(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t) = I_p(t)\Pi_p(t),$$
$$\widehat{F}_2(t) = \mathrm{E}[F_2(t)|data] \tag{2.31}$$
$$= \Phi_{ps}(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t) = 1 - I_{ps}(t)\Pi_{ps}(t).$$

$\widehat{F}_1(t)$, and $\widehat{F}_2(t)$ are the nonparametric estimators of $F_1(t)$, and $F_2(t)$, respectively, based on posterior means.

As in Theorem 5, it is straightforward to express the nonparametric estimator of $F_3(t)$. In the next section, we extend the estimators to a general case of $m \geq 4$.

2.3   Bayesian estimator for $m \geq 4$

The extension of the nonparametric Bayesian estimator for SPS and PSS, given in Section 2.2, is based on rewriting the system representation in a proper simplified version of the general case ($m \geq 4$) to the one given with $m = 3$, which has a solution given in Theorem 6. Considering the SPS and the PSS presented in Fig. 2.1, we specify how to rewrite the system representation and estimation of their components reliabilities in the following.

We provided how to estimate Y_1 (for the SPS), and Z_1 (for the PSS), because the reliability estimation of the other components are straightforward once these two are given. The idea of the extension is to represent the systems in a simple version with three components (Figures 1.3a and
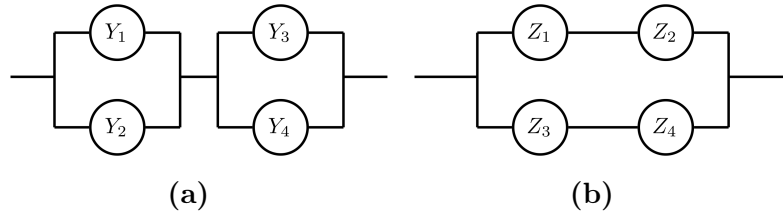
Figure 2.1: *(a) SPS (b) PSS*

1.4a). In this case, to estimate the reliability of Y_1, we use the SPS solution considering $X_1 = \max(Y_3, Y_4)$, $X_2 = Y_1$, and $X_3 = Y_2$ (Figure 2.2a); and for the estimation of Z_1, we use the PSS solution considering $X_1 = \min(Y_3, Y_4)$, $X_2 = Y_1$, and $X_3 = Y_2$ (Figure 2.2b). It must be noted that other more complex systems can also be considered, but the task is only to simplify the representation of the system as one of either the PSS or SPS given in Figures 1.3a and 1.4a.
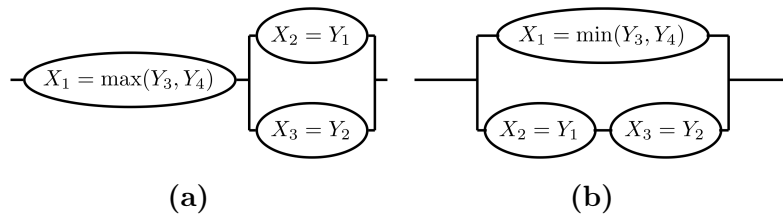
Figure 2.2: *(a) SPS (b) PSS*

Furthermore, both the classes (SPS and PSS) are important so as to have a more general solution, because we have the restriction that two different components cannot have the same failure time, which in turn would result in different representations giving more options to the reliability estimation problem. Considering the PSS given in Figure 2.1b, we can write their SPS representation as that presented in Figure 2.3. The component's reliability of the original PSS (Figure 2.1b) can be estimated using the PSS result of Theorem 6, which has a simple solution. However,

as the SPS representation (Figure 2.3) has some components repeated, the SPS result of Theorem 6 is not applicable. Thus, the solutions for both SPS and PSS are important and can be used in different situations.
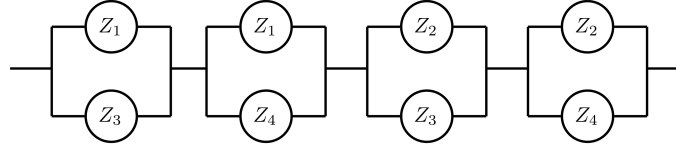
Figure 2.3: *The SPS representation of the PSS in Figure 2.1b.*

2.4   Simulated datasets

This section presents two examples to demonstrate the estimation steps and to show the quality of the Bayesian nonparametric estimator. The estimation steps for the PSS are very similar to those for the SPS, and for the sake of brevity, we have omitted them. The estimation steps for SPS are as follows.

1. *Defining priors:* The prior measures $(\alpha_1, \alpha_2, \alpha_3)$ are prior guesses of the SDF (F_1^*, F_2^*, F_3^*), but it is not simple to elicit these measures. It is easier to elicit the priors for the DF (F_1, F_2, F_3), and use (2.13) for the SPS to evaluate the prior measures (for PSS we can use (2.14)). We chose the exponential distribution with mean 1 as the prior guess for each of the three components DF. By evaluating the prior measures using (2.13), we have $\alpha_1(0, v] = (e^{-3v} - 3e^{-2v} + 2)/3$, and $\alpha_2(0, v] = \alpha_3(0, v] = (2e^{-3v} - 3e^{-2v} + 1)/6$. Note that this prior is not very informative because the measure of the whole parameter space is only one $(\alpha_1(\Omega) + \alpha_2(\Omega) + \alpha_3(\Omega) = 1)$. Also, we

have that $d\alpha_1(0, v] = 2e^{-2v} - e^{-3v}dv$, and $d\alpha_2(0, v] = d\alpha_3(0, v] = e^{-2v} - e^{-3v}dv$.

2. *Obtaining Posteriors:* The posterior processes for the SDF functions are $\mathcal{D}((e^{-3t} - 3e^{-2t} + 2)/3 + nF_{1n}^*(t), (2e^{-3t} - 3e^{-2t} + 1)/6 + nF_{2n}^*(t), (2e^{-3t} - 3e^{-2t} + 1)/6 + nF_{3n}^*(t))$; and from (2.17), we have

$$
\begin{aligned}
\widehat{F}_1^*(t) &= \frac{nF_{1n}^*(t) + (e^{-3t} - 3e^{-2t} + 2)/3}{n + 1}, \\
\widehat{F}_2^*(t) &= \frac{nF_{2n}^*(t) + (2e^{-3t} - 3e^{-2t} + 1)/6}{n + 1}, \\
\widehat{F}_3^*(t) &= \frac{nF_{3n}^*(t) + (2e^{-3t} - 3e^{-2t} + 1)/6}{n + 1},
\end{aligned}
$$

which are the estimators of SDF.

3. *Computing system's reliability:* (2.18) provides the estimator of the system distribution function. For the prior defined earlier, we have

$$
\widehat{F}(t) = \frac{nF_n^*(t) + e^{-3t} - 2e^{-2t} + 1}{n + 1}.
$$

4. *Computing components reliabilities:* Theorem 6 gives the estimators of the components DF. Using (2.28), we obtain the estimate for component 1 DF; and from (2.29), we obtain the estimate for component 2 DF. For Component 3 DF, we substitute $d\alpha_2(0, v]$ by $d\alpha_3(0, v]$ (in the integral part I_{sp}), and d_{2i} by d_{3i} (in the product part Π_{sp}) in (2.29). Also, the integral part of the estimator can be solved by using a numerical procedure, such as the Simpson's rule. For more details and other numerical integration methods, see Davis e Rabinowitz (1984).

2.4.1   Simulated dataset 1

We obtained 100 observations of SPS presented in Figure 2.1a, where all the components had gamma distributions; the first component (Y_1) had a mean of 4 and a standard deviation (SD) of 2.83, the second component (Y_2) had a mean of 6 and a SD of 4.9, the third component (Y_3) had a mean of 8 and a SD of 5.67, and the fourth component (Y_4) had a mean of 3 and a SD of 2.45. The Bayesian estimators are based on $(T_1, \delta_1), \ldots, (T_{100}, \delta_{100})$. The simulated values are listed in Table A.1 in Appendix A.

To estimate components 1–4, we rewrote the representation of the system as follows. For component 1, we considered that $X_1 = \max(Y_3, Y_4)$, $X_2 = Y_1$, and $X_3 = Y_2$; then $\widehat{F}_{Y_1}(t) = \widehat{F}_2(t)$, where $\widehat{F}_{Y_1}$ is the DF estimate of component 1, and $\widehat{F}_2$ is the proposed estimator for SPS (2.29). In a similar way, for component 2, we considered $X_1 = \max(Y_3, Y_4)$, $X_2 = Y_2$, and $X_3 = Y_1$; for component 3, we considered $X_1 = \max(Y_1, Y_2)$, $X_2 = Y_3$, and $X_3 = Y_4$; and for component 4, we considered $X_1 = \max(Y_1, Y_2)$, $X_2 = Y_4$, and $X_3 = Y_3$ (see Section 2.3). We found that the proportions of censored data for components 1–4 are 77%, 64%, 73%, and 86%, respectively.

Figure 2.4 presents the estimates of the five distribution functions associated with components 1–4 and the system. In all plots, the *true* distribution functions (dashed lines) and the prior mean (dashed-dot line) are also illustrated. The conditional reliabilities of the components relative to the system are $\widehat{\rho}_1 \cong 0.2294$, $\widehat{\rho}_2 \cong 0.3581$, $\widehat{\rho}_3 \cong 0.2690$, and $\widehat{\rho}_4 \cong 0.1403$.
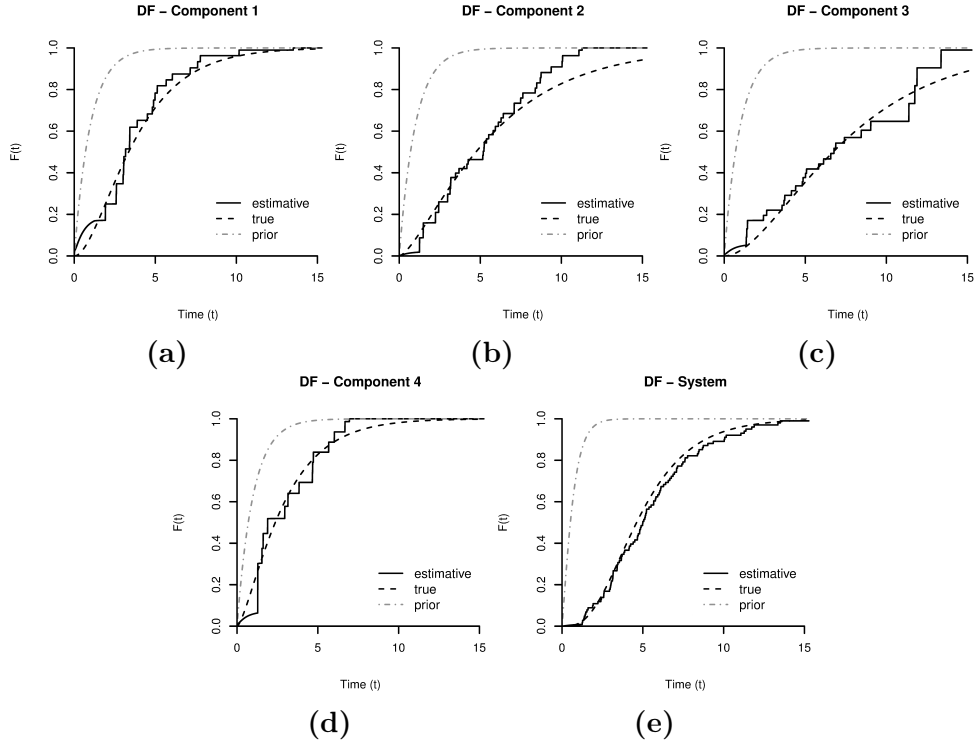
Figure 2.4: *Estimates for the Example 2.4.1.*

2.4.2   Simulated dataset 2

In this example, we considered a PSS represented in Figure 2.1b. One of the components had the distribution function of a mixture of an exponential distribution and a discrete distribution, with positive probability to fail at times 1 and 3, and is given by

$$F(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.6(1 - e^{-t/4}), & \text{if } t < 1, \\ 0.6(1 - e^{-t/4}) + 0.25, & \text{if } t < 3, \\ 0.6(1 - e^{-t/4}) + 0.4, & \text{if } t \geq 3. \end{cases} \tag{2.32}$$

We obtained 100 observed systems, where the first component (Z_1) had a gamma distribution with a mean of 4 and a SD of 2.83, the second

component (Z_2) had a Weibull distribution with a mean of 4.51 and a
SD of 3.06, the third component (Z_3) had a mixture of an exponential
and a discrete distribution (2.32) with a mean of 3.1 and a SD of 3.34,
and the fourth component (Z_4) had a log-normal distribution with a
mean of 4.59 and a SD of 2.45. The Bayesian estimators are based on
$(T_1, \delta_1), \ldots, (T_{100}, \delta_{100})$. The simulated values are listed in Table A.2 in
Appendix A.

To estimate the parameters for components 1 through 4, we rewrote
the representation of the system as follows. For component 1, we consid-
ered that $X_1 = \min(Z_3, Z_4)$, $X_2 = Z_1$, and $X_3 = Z_2$; then, $\widehat{F}_{Z_1}(t) = \widehat{F}_2(t)$,
where $\widehat{F}_{Z_1}$ is the DF estimate of component 1, and $\widehat{F}_2$ is the proposed es-
timator for PSS (2.31). In a similar way, for component 2, we considered
$X_1 = \min(Z_3, Z_4)$, $X_2 = Z_2$, and $X_3 = Z_1$; for component 3, we consid-
ered $X_1 = \min(Z_1, Z_2)$, $X_2 = Z_3$, and $X_3 = Z_4$; and for component 4,
we considered $X_1 = \min(Z_1, Z_2)$, $X_2 = Z_4$, and $X_3 = Z_3$ (see Section
2.3). We found that the proportion of the censored data for components
1 through 3 is 71%, and that for component 4 is 87%.

Figure 2.5 presents the estimates of the five distribution functions:
components 1 through 4 and the system. In all the plots, the *true* distri-
bution functions (dashed lines) and the prior mean (dashed-dot line) are
also illustrated. The conditional reliabilities of the components relative to
the system are $\widehat{\rho}_1 \cong 0.2887$, $\widehat{\rho}_2 \cong 0.2887$, $\widehat{\rho}_3 \cong 0.2887$, and $\widehat{\rho}_4 \cong 0.1303$.
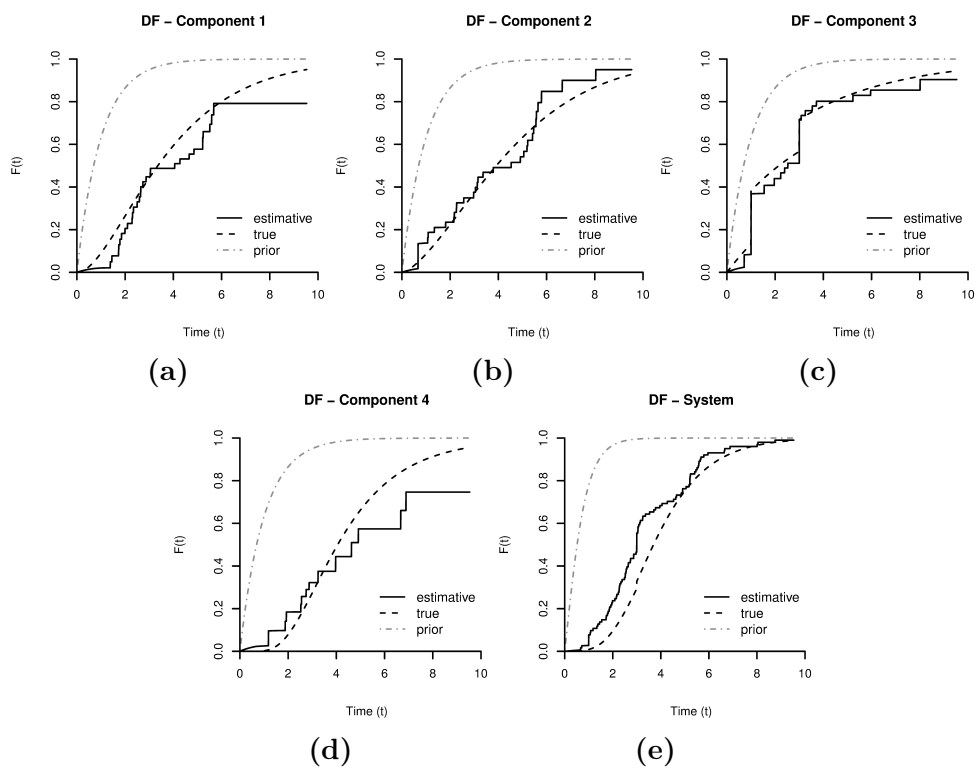
Figure 2.5: *Estimates for the Example 2.4.2.*

Chapter 3

Weibull model

We present a Bayesian parametric approach for inferences about components' reliabilities in coherent systems when system failure time and status of each component in moment of system failure are available. For that, the method does not need the assumption of identically distributed components lifetimes. The main assumption is that components' lifetime are mutually independent.

We assume the three-parameter Weibull distribution as the component failure time distribution, presented in Section 3.1, a very general distribution that can approximate most of the lifetimes distributions. The Weibull model is presented in Section 3.2. In Section 3.3 simulated datasets are considered and in Section 3.4 a real dataset is considered in order to show the applicability of presented model.

3.1   Three-parameter Weibull distribution

The Weibull distribution was thus named after Waloddi Weibull presented it to the world scientific community in 1951 (Weibull, 1951), but previously considered this distribution in data of resistance of materials (Weibull, 1939).

The Weibull distribution is undoubtedly one of the most popular models in statistics because of its ability to fit data from a variety of areas, from survival data to weather data or observations made in economics, hydrology, biology or engineering (Rinne, 2008).

The three-parameter Weibull reliability function is given by

$$R(t \mid \boldsymbol{\theta}_j) = \exp\left[-\left(\frac{t - \mu_j}{\eta_j}\right)^{\beta_j}\right], \tag{3.1}$$

for $t > 0$, where $\boldsymbol{\theta}_j = (\beta_j, \eta_j, \mu_j)$ and $\beta_j > 0$ (shape), $\eta_j > 0$ (scale) and $0 < \mu_j < t$ (location). The three-parameter Weibull density function is

$$f(t \mid \boldsymbol{\theta}_j) = \frac{\beta_j}{\eta_j}\left(\frac{t - \mu_j}{\eta_j}\right)^{\beta_j - 1} \exp\left[-\left(\frac{t - \mu_j}{\eta_j}\right)^{\beta_j}\right]. \tag{3.2}$$

The Weibull distribution has characteristics that make this distribution a great candidate to model components lifetimes. One of them is that variation of parameter values implies changes in both distribution shape and hazard rates. We can have increasing, decreasing and constant failure rates in this family of Weibull distributions Rinne (2008). Besides, the three-parameter Weibull family is a very rich family since most real situations will have random aspects that can be represented by an element of the family.

The Weibull distribution with two parameters ($\mu_j = 0$) is the most celebrated case in the literature. However, the location parameter μ_j that represents the baseline lifetime has an important meaning in reliability and survival analysis. In reliability, a component under test may be used at an earlier moment and the beginning of the test is not necessarily the beginning of the component's life. In medicine for instance, a patient may have the disease before the onset medical appoitment. Not taking account this initial time can underestimates the other parameters. Clearly, in a situation of new component testing μ_j may be 0.

3.2   Weibull model

We assume that $X_1, \ldots, X_m$ are mutually independent. This assumption is necessary under the considered approach, as it is proved in a Theorem at Rodrigues *et al.* (2018).

Considering the reliability and density functions in (3.1) and (3.2) in the likelihood function presented in (1.3), the likelihood function for the Weibull parameters $\boldsymbol{\theta}_j$ is given by

$$
\mathrm{L}(\boldsymbol{\theta}_j \mid \mathbf{l}_j, \mathbf{u}_j) = \prod_{i=1}^{n} \left\{ \frac{\beta_j}{\eta_j} \left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j - 1} \exp\left[-\left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \right\}^{\mathrm{I}_{\{l_{ji} = u_{ji}\}}}
$$
$$
\left\{ \exp\left[-\left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] - \exp\left[-\left(\frac{u_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \right\}^{1 - \mathrm{I}_{\{l_{ji} = u_{ji}\}}}, \quad (3.3)
$$

where $\mathrm{I}_{\{TRUE\}} = 1$ or $\mathrm{I}_{\{FALSE\}} = 0$, $\mathbf{l_j} = (l_{j1}, \ldots, l_{jn})$ and $\mathbf{u_j} = (u_{j1}, \ldots, u_{jn})$.

The posterior density of $\boldsymbol{\theta}_j = (\beta_j, \eta_j, \mu_j)$ comes out to be

$$
\pi(\beta_j, \eta_j, \mu_j \mid \mathbf{l}_j, \mathbf{u}_j) \propto
$$

$$\pi(\beta_j, \eta_j, \mu_j) \prod_{i=1}^{n} \left\{ \left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j - 1} \frac{\beta_j}{\eta_j} \exp\left[-\left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \right\}^{I_{\{l_{ji} = u_{ji}\}}}$$

$$\times \left\{ \exp\left[-\left(\frac{l_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] - \exp\left[-\left(\frac{u_{ji} - \mu_j}{\eta_j} \right)^{\beta_j} \right] \right\}^{1 - I_{\{l_{ji} = u_{ji}\}}} \quad (3.4)$$

for which $\pi(\beta_j, \eta_j, \mu_j)$ is the prior density of $\boldsymbol{\theta}_j = (\beta_j, \eta_j, \mu_j)$ that we consider to be:

$$\pi(\beta_j, \eta_j, \mu_j) = \frac{1}{\eta_j} \frac{1}{\beta_j}. \quad (3.5)$$

Even (3.5) being not a proper prior − its integral is not finite − the posterior density in Equation (3.4) is proper, as stated by the following result (Rodrigues *et al.*, 2018).

Theorem 7 *Let a class of non-informative prior given by*

$$\pi(\beta_j, \eta_j, \mu_j) = \frac{1}{\eta_j \beta_j^b}, \quad b \geq 0.$$

Although, for $b \geq 0$, $n = 1$ and the existence of a failure, the posterior in (3.4) is not proper, for $n > 1$, the posterior in (3.4) is proper.

The proof of this result can be seen at Rodrigues *et al.* (2018). The importance of the above result is that one can perform Bayesian inferences even with low prior information.

Because the posterior density (3.4) has not a closed form, statistical inferences about the parameters can rely on Markov-Chain Monte-Carlo (MCMC) simulations. Here we consider an adaptive-Metropolis-Hasting algorithm with a multivariate distribution (Haario *et al.*, 2005).

Discarding burn-in (first generated values discarded to eliminate the effect of the assigned initial values for parameters) and jump samples

(spacing among generated values to avoid correlation problems) a sample of size n_p from the joint posterior distribution of $\boldsymbol{\theta}_j$ is obtained. For the jth component, the sample from the posterior can be expressed as $(\beta_j^{(1)}, \beta_j^{(2)}, \ldots, \beta_j^{(n_p)})$, $(\eta_j^{(1)}, \eta_j^{(2)}, \ldots, \eta_j^{(n_p)})$ and $(\mu_j^{(1)}, \mu_j^{(2)}, \ldots, \mu_j^{(n_p)})$. Consequently, posterior quantities of reliability function $R(t \mid \boldsymbol{\theta}_j)$ can be easily obtained (Robert e Casella, 2010). For instance, the posterior mean of the reliability function is

$$\mathrm{E}\Big[R(t \mid \boldsymbol{\theta}_j) \mid \mathbf{l}_j, \mathbf{u}_j\Big] = \frac{1}{n_p} \sum_{k=1}^{n_p} R\Big(t \mid \boldsymbol{\theta}_j^{(k)}\Big), \quad \text{for each } t > 0. \qquad (3.6)$$

3.3   Simulated datasets

3.3.1   Parallel system simulated data

A sample of $n = 30$ parallel systems with $m = 3$ components was generated. In the generation process, X_1 was generated from a log-normal distribution with mean 5.5 and variance 7, X_2 from a Weibull distribution with mean 4 and variance 12 and X_3 was generated from three-parameter Weibull distribution with mean 5 and variance 3. In a parallel structure, the system works when at least one component works, so the system lifetime is given by $T = \max\{X_1, X_2, X_3\}$. The generated data are presented in Table B.1, Appendix B. In this structure, when a given component is not the last to fail, it is censored to the left. In observed sample, component 1 presents 60% of left-censored data and components 2 and 3 are 70% left-censored data each.

To obtain posterior quantities based on posterior density (3.4) through

MCMC simulations, 20,000 samples were generated, in which the first 10,000 of them were discarded as burn-in samples and jump of size 10 was chosen to avoid correlation problems. Consequently, samples of size $n_p = 1,000$ were obtained of each posterior quantitites. The chains convergence was monitored for good convergence results to be obtained (Robert e Casella, 2010). Posterior measures of Weibull parameters β_j, η_j and μ_j, for $j = 1, 2, 3$, are presented in Table 3.1 and posterior measures of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 3.2. The posterior means of reliability functions can be visualized in Figure 3.1 besides the empirical 95% HPD intervals (Highest Posterior Density). The reliability posterior means are close to the true reliability curves, in general. Even for values of t that the posterior mean is more distant from the true values, the lower or upper limit of HPD interval is very close to the true curve.

Table 3.1: *Posterior measures of Weibull parameters for components involved in parallel system.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Component 1				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_1	0.297	0.636	0.780	0.817	0.959	2.053	0.258	0.360 - 1.359
η_1	0.258	1.619	2.297	2.495	3.144	6.574	1.215	0.454 - 5.062
μ_1	0.014	1.794	2.386	2.184	2.783	3.131	0.764	0.465 - 3.131
				Component 2				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_2	0.132	0.612	0.874	0.935	1.182	2.557	0.422	0.272 - 1.774
η_2	0.016	1.522	2.598	2.656	3.713	7.318	1.423	0.156 - 5.078
μ_2	0.005	0.709	1.656	1.642	2.581	3.156	0.998	0.109 - 3.137
				Component 3				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_3	0.164	0.734	1.005	1.157	1.449	3.635	0.581	0.340 - 2.421
η_3	0.124	1.008	1.557	1.883	2.528	5.405	1.142	0.250 - 4.328
μ_3	0.003	1.856	2.630	2.295	2.971	3.156	0.862	0.365 - 3.156

Table 3.2: *Posterior measures of reliability functions for some values of t in parallel components.*

t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
			Component 1					
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
0.50	0.899	1.000	1.000	0.998	1.000	1.000	0.009	0.997 - 1.000
10.00	0.010	0.051	0.073	0.083	0.106	0.271	0.043	0.015 - 0.170
16.50	0.000	0.006	0.014	0.022	0.030	0.154	0.022	0 - 0.068
			Component 2					
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
1.00	0.565	0.975	1.000	0.971	1.000	1.000	0.061	0.831 - 1.000
11.50	0.000	0.019	0.037	0.045	0.063	0.244	0.035	0 - 0.112
17.00	0.000	0.002	0.008	0.016	0.023	0.146	0.022	0 - 0.063
			Component 3					
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
0.30	0.958	1.000	1.000	1.000	1.000	1.000	0.002	0.998 - 1.000
9.00	0.000	0.007	0.017	0.024	0.034	0.182	0.025	0 - 0.076
12.50	0.000	0.000	0.002	0.007	0.007	0.140	0.013	0 - 0.032

(a) *Component 1*   **(b)** *Component 2*   **(c)** *Component 3*
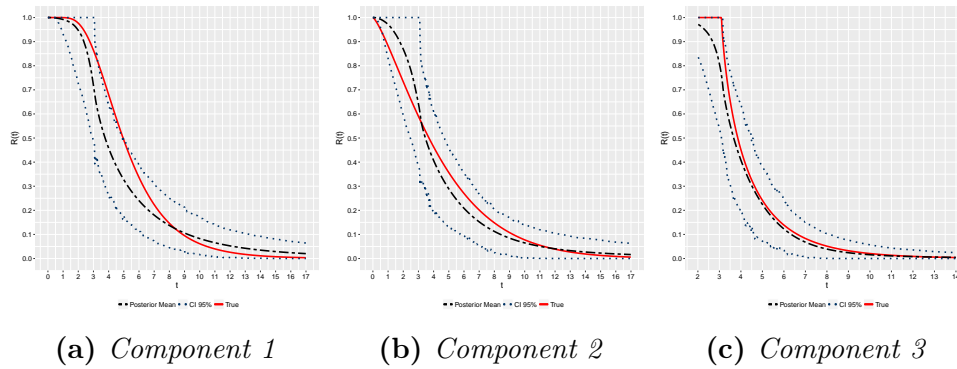
Figure 3.1: *Generating reliability functions, posterior means and 95% HPD intervals (CI 95%) for the components involved in parallel structure with three components.*

3.3.2   2-out-of-3 system simulated data

A sample of $n = 50$ 2-out-of-3 systems was generated. In the generation process, X_1 was generated from a Weibull distribution with mean 15 and variance 8, X_2 from a gamma distribution with mean 18 and variance 12 and X_3 from a lognormal distribution with mean 20 and variance 10. In a 2-out-of-3 structure, the system stops working when the second component fails, so the system lifetime is given by $T = \max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\}$. The generated data are presented in Table B.2, Appendix B.

In this structure, when a given component is not the second component to fail, it is censored either to the right or to the left side. In observed sample, component 1 presents 76% of censored data (70% to the left and 6% to the right), component 2 presents 48% of censored data (22% to the left and 26% to the right) and component 3 presents 76% of censored data (8% to the left and 68% to the right).

To obtain posterior quantities based on posterior density (3.4) through MCMC simulations, 20,000 samples were generated, in which the first 10,000 of them were discarded as burn-in samples and jump of size 10 was chosen to avoid correlation problems. Consequently, samples of size $n_p = 1,000$ were obtained of each posterior quantitites. The chains convergence was monitored for good convergence results to be obtained (Robert e Casella, 2010). Posterior measures of Weibull parameters β_j, η_j and μ_j, for $j = 1, 2, 3$, are presented in Table 3.3 and posterior measures of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 3.4. The posterior means of reliability functions can be visualized in Figure 3.2 besides the empirical 95% HPD intervals. The reliability posterior means are close to the true reliability curves mainly for components 1 and 3. For component 2, even for values of t that the posterior mean is more distant from the true values of the reliability, the upper limit of HPD interval is very close to the true curve.

Table 3.3: *Posterior measures of Weibull parameters for 2-out-of-3 components.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Component 1				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_1	0.552	1.480	2.281	2.619	3.562	7.297	1.394	0.601 - 5.376
η_1	1.232	3.819	6.262	7.260	10.240	16.540	4.088	1.651 - 15.101
μ_1	0.001	5.443	9.328	8.416	11.900	13.310	3.950	0.708 - 13.212
				Component 2				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_2	1.006	2.268	3.169	3.622	4.606	9.703	1.693	1.442 - 7.192
η_2	4.107	6.409	8.445	9.612	12.190	19.520	3.968	4.364 - 17.503
μ_2	0.035	6.255	9.974	8.816	12.020	13.350	3.813	1.077 - 13.347
				Component 3				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_3	0.922	2.381	3.396	3.992	5.243	12.540	2.028	1.318 - 8.005
η_3	6.135	9.210	11.810	12.750	15.740	24.440	4.107	7.122 - 20.820
μ_3	0.001	5.656	9.752	8.666	12.290	13.350	4.035	0.837 - 13.353

Table 3.4: *Posterior measures of reliability functions for some values of t in 2-out-of-3 components.*

t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Component 1				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
7.70	0.811	0.997	1.000	0.994	1.000	1.000	0.017	0.959 - 1.000
15.00	0.204	0.405	0.471	0.471	0.532	0.725	0.090	0.307 - 0.637
20.00	0.002	0.023	0.038	0.043	0.056	0.191	0.027	0.003 - 0.094
				Component 2				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
9.45	0.955	0.999	1.000	0.998	1.000	1.000	0.004	0.990 - 1.000
16.00	0.520	0.665	0.705	0.703	0.745	0.858	0.058	0.588 - 0.807
22.00	0.001	0.028	0.048	0.055	0.074	0.259	0.037	0.003 - 0.131
				Component 3				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	HPD 95%
14.00	0.830	0.955	0.973	0.967	0.985	0.999	0.024	0.919 - 0.997
22.00	0.038	0.205	0.277	0.288	0.368	0.654	0.113	0.067 - 0.492
26.00	0.000	0.007	0.035	0.067	0.100	0.482	0.079	0.000 - 0.227

(a) *Component 1*   (b) *Component 2*   (c) *Component 3*
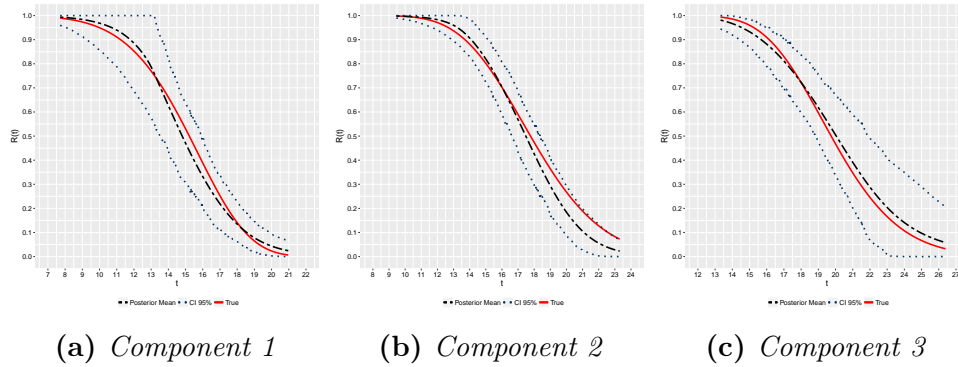
Figure 3.2: *Generating reliability functions, posterior means and 95% HPD intervals (CI 95%) for the components involved in* 2-*out-of-*3 *structure.*

3.3.3   Bridge system simulated data

A sample of $n = 100$ bridge systems as representation in Figure 1.5 was simulated. In the generation process, X_1 was generated from a Weibull distribution with mean 17 and variance 8, X_2 from a log-normal distribution with mean 16 and variance 22, X_3 from a log-normal distribution with mean 15 and variance 15, X_4 from a gamma distribution with mean 15 and variance 6 and X_5 from a gamma distribution with mean 20 and variance 12. In a bridge structure, the system stops depending on a combination of components failures, so the system lifetime is given by $T = \max\{\min\{X_1, X_4\}, \min\{X_2, X_5\}, \min\{X_1, X_3, X_5\}, \min\{X_2, X_3, X_4\}\}$. The generated data are presented in Table B.3, in Appendix B. In observed sample, component 1 presents 83% of censored data (31% to the left and 52% to the right), component 2 presents 71% of censored data (49% to the left and 22% to the right), component 3 presents 89% of censored data (64% to the left and 25% to the right), component 4 presents 74% of censored data (56% to the left and 18% to the right) and com-

ponent 5 presents 83% of censored data (11% to the left and 72% to the right).

To obtain posterior quantities based on posterior density (3.4) through MCMC simulations, 30,000 samples were generated, in which the first 10,000 of them were discarded as burn-in samples and jump of size 20 was chosen to avoid correlation problems. Consequently, samples of size $n_p = 1,000$ were obtained of each posterior quantitites. The chains convergence was monitored for good convergence results to be obtained (Robert e Casella, 2010). Posterior measures of Weibull parameters β_j, η_j and μ_j, for $j = 1, \ldots, 5$, are presented in Table 3.5 and posterior measures of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 3.6. The posterior means of reliability functions can be visualized in Figure 3.3 besides the empirical 95% HPD intervals. The reliability posterior means are close to the true reliability curves for all components, mainly for component 2. Even for values of t that the posterior mean is more distant from the true values, the lower or upper limit of HPD interval is very close to the true curve.

Table 3.5: *Posterior measures of Weibull parameters for the components in the bridge structure.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Component 1				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_1	0.993	3.561	4.523	4.611	5.560	8.665	1.409	2.061 - 7.335
η_1	7.236	11.766	14.245	14.089	16.596	19.790	2.971	8.861 - 18.917
μ_1	0.010	1.909	4.145	4.408	6.708	10.801	2.913	0.010 - 9.436
				Component 2				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_2	1.163	2.219	2.785	2.829	3.408	5.509	0.822	1.333 - 4.343
η_2	5.009	9.398	11.644	11.742	14.328	17.710	3.066	6.035 - 16.791
μ_2	0.001	2.563	5.043	4.955	7.264	10.684	2.924	0.008 - 9.78
				Component 3				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_3	0.568	1.272	1.608	1.830	2.174	4.951	0.776	0.772 - 3.534
η_3	2.607	5.124	6.389	7.558	9.088	16.909	3.241	3.534 - 14.742
μ_3	0.018	6.399	9.036	7.907	10.273	10.899	3.029	1.282 - 10.899
				Component 4				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_4	1.203	2.167	2.641	3.039	3.605	8.107	1.197	1.403 - 5.491
η_4	4.137	5.280	6.341	7.464	8.811	16.534	2.916	4.284 - 14.346
μ_4	0.008	7.096	9.495	8.320	10.468	10.897	2.813	1.674 - 10.897
				Component 5				
	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
β_5	1.034	2.570	3.342	3.597	4.450	7.778	1.289	1.562 - 6.098
η_5	9.044	12.236	14.542	15.060	17.716	24.581	3.373	9.823 - 21.195
μ_5	0.028	4.185	7.301	6.662	9.555	10.894	3.231	0.777 - 10.894

Table 3.6: *Posterior measures of components' reliability functions for some values of t in the bridge structure.*

t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Component 1				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
8.95	0.922	0.990	0.995	0.993	0.999	1.000	0.008	0.976 - 1.000
15.00	0.598	0.727	0.758	0.757	0.790	0.869	0.046	0.671 - 0.845
22.00	0.001	0.031	0.059	0.070	0.098	0.346	0.052	0.001 - 0.174
				Component 2				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
6.50	0.927	0.988	0.998	0.992	1.000	1.000	0.012	0.967 - 1.00
18.00	0.142	0.233	0.259	0.261	0.286	0.404	0.041	0.190 - 0.351
32.00	0.000	0.000	0.000	0.001	0.000	0.012	0.002	0 - 0.004
				Component 3				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
7.50	0.767	0.992	1.000	0.987	1.000	1.000	0.030	0.92 - 1.000
13.00	0.348	0.576	0.626	0.624	0.677	0.834	0.074	0.485 - 0.769
24.00	0.000	0.010	0.021	0.028	0.036	0.174	0.025	0 - 0.079
				Component 4				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
9.00	0.933	0.996	1.000	0.995	1.000	1.000	0.011	0.971 - 1.000
14.00	0.427	0.610	0.651	0.649	0.689	0.801	0.057	0.542 - 0.756
20.00	0.001	0.013	0.022	0.026	0.036	0.145	0.019	0.001 - 0.063
				Component 5				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
12.00	0.896	0.967	0.978	0.975	0.986	0.998	0.015	0.947 - 0.997
20.00	0.295	0.455	0.503	0.503	0.550	0.697	0.073	0.371 - 0.654
29.50	0.000	0.002	0.012	0.032	0.041	0.319	0.048	0 - 0.142

(a) *Component 1*   (b) *Component 2*   (c) *Component 3*

(d) *Component 4*   (e) *Component 5*
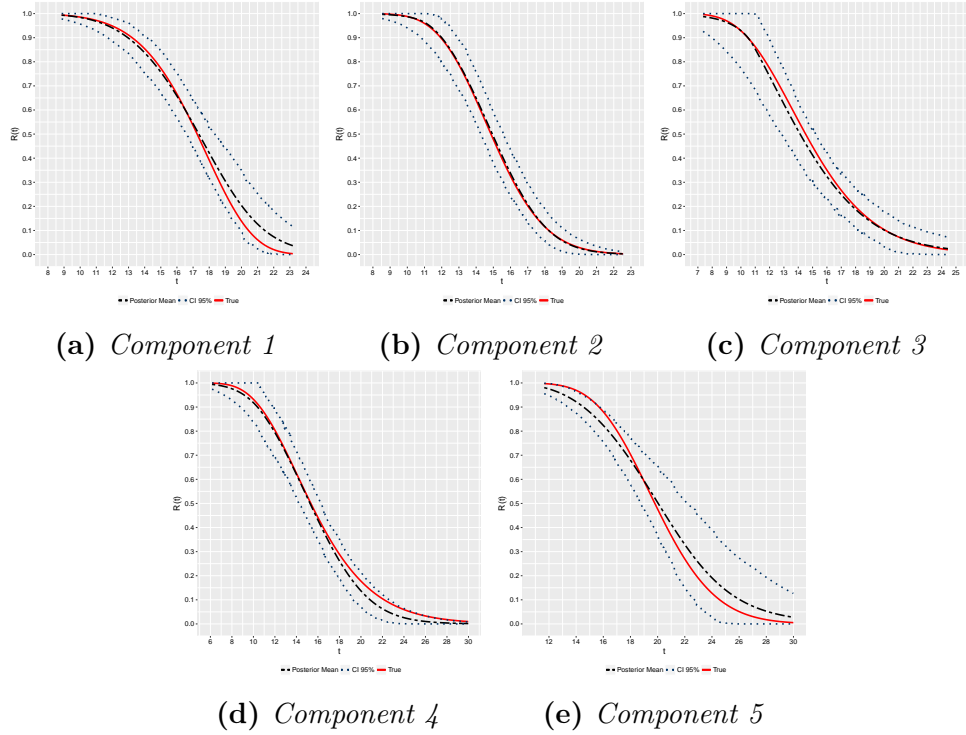
Figure 3.3: *Generating reliability functions, posterior means and 95% HPD intervals (CI 95%) for the components involved in the bridge structure.*

3.4   Device-G dataset

A dataset presented by Meeker e Escobar (2014) is considered. A sample of $n = 30$ units were installed in typical service environments. Cause of failure information was determined for each unit that failed. Mode S failures were caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms. Mode W failures were caused by normal product wear that began to appear after many cycles of use. The causes W and S competed to be responsible for device-G failure, i.e., it is a system with 2 components in series. The observed dataset is presented in Table B.5 (Appendix B), in which 23.33% of observed system failure because of component W (component S is right-

censored), 50% of systems failed because of component S failure (component W is right-censored) and the remaining 26.67% were censored systems and for them, both components are right-censored observations.

To obtain posterior quantities related to the posterior distribution of $\boldsymbol{\theta} = (\beta, \eta, \mu)$ from posterior distribution in (3.4) through MCMC simulations, we discarded the first 10,000 as burn-in samples and jump of size 10 to avoid correlation problems, obtaining a sample of size $n_p = 1,000$. The chains convergence was monitored for all simulation scenarios and good convergence results were obtained.

Table 3.7 lists the posterior quantities for the parameters of shape (β), scale (η) and location (μ). The posterior measures of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 3.8 and posterior means of reliability functions can be visualized in Figure 3.4 besides the empirical 95% HPD intervals.

Table 3.7: *Posterior measures of Weibull parameters for Device-G dataset.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Componente W ($j = 1$)				
β_1	1.440	3.329	4.117	4.246	4.999	9.931	1.314	1.895 - 6.875
η_1	265.166	315.879	335.632	341.645	360.237	581.516	37.428	283.810 - 418.451
μ_1	0.001	0.479	1.004	0.994	1.518	1.999	0.587	0.004 - 1.894
				Componente S ($j = 2$)				
β_2	0.271	0.521	0.613	0.620	0.709	1.296	0.148	0.372 - 0.913
η_2	102.848	238.267	294.820	311.932	367.790	722.754	98.473	163.956 - 535.053
μ_2	0.001	0.881	1.558	1.357	1.871	2.000	0.593	0.204 - 2.000

Table 3.8: *Posterior measures of reliability functions for some values of t in Device-G dataset.*

				Componente W				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
11.046	0.996	1.000	1.000	1.000	1.000	1.000	0.000	0.999 - 1.000
150.230	0.761	0.938	0.966	0.955	0.982	0.999	0.039	0.875 - 0.999
250.023	0.423	0.687	0.748	0.741	0.804	0.940	0.086	0.58 - 0.896
				Componente S				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
14.988	0.573	0.811	0.861	0.850	0.898	0.984	0.063	0.728 - 0.956
120.191	0.353	0.512	0.564	0.564	0.614	0.778	0.071	0.437 - 0.702
250.071	0.187	0.359	0.404	0.408	0.456	0.600	0.067	0.275 - 0.528

(a) *Component W*   (b) *Component S*
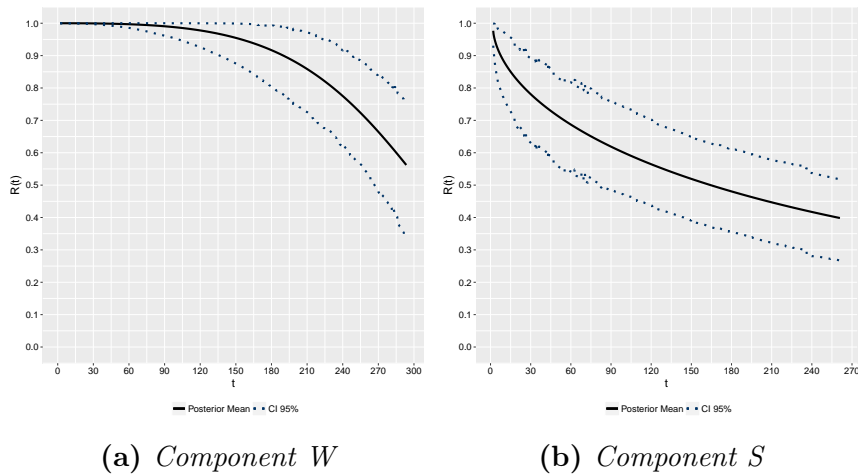
Figure 3.4: *Generating reliability functions, posterior means and 95% HPD intervals (CI 95%) for the components involved in Device-G dataset.*

Chapter 4

Masked data

Chapters 2 and 3 considered the problem of component estimation in coherent systems under the Bayesian paradigm in nonparametric and parametric approaches, respectively. In both cases the status of each component at the time of system failure is considered to be known. In some situation, however, the component that causes the system failure is not identified exactly and can only be narrowed down to a smaller set of components, known as masked data failure cause.

Flehinger *et al.* (2002) presents a masked data problem in which consists of computer hard-drives failure times monitored over a period of 4 years. There were three possible causes of failure competing to be computer hard-drives failures: eletronic hard, head flyability and head/disc magnetics. For some units, the real cause of hard-drive failure is not identified only that it belongs to a subset of components candidates to be system failure cause.

The masked data problem formulation is developed and a Bayesian three-parameter Weibull model for marginal components' reliabilities in

masked data scenario is presented. This chapter is based on the work of Rodrigues *et al.* (2017) and we suggest seeing this article for more details.

In Section 4.1 masked data problem is discussed. Weibull model for masked data scenario is presented in 4.2. Section 4.3 presents simulated datasets for some system structures and Section 4.4 presents computer hard-drives dataset solution.

4.1   Masked data scenario

Consider again that a random sample of $n = 10$ 2-out-of-3 systems (Figure 1.7) is observed. Unlike the previous situations, the component whose failure causes the system failure is not identified for all units. A set of components index, say s, which includes the component responsible for the system failure is specified. The data are presented in Table 4.1. For instance, system ID=4 failed at time 2.55 but the component that cause the system failure is not identified, it is just known that the failure of component 2 or the failure of component 3 caused system failure, once $s = \{2, 3\}$. If your interest is inference about component 2, por example, it is not known whether the failure of component 2 produced the failure of the system or if component 2 is left-censored at time 2.55.

Note that for system ID=5 there is only one component in set s, $s = \{1\}$. Thus, it is not a case of masked data, in which component 1 failed at time 1.89 and its failure caused the system to fail.

Table 4.1: *Observed data of $n = 10$ 2-out-of-3 systems in masked data scenario.*

System ID	t	s
1	1.95	{1,3}
2	2.09	{1,2}
3	3.56	{2}
4	2.55	{2,3}
5	1.89	{1}
6	3.01	{2,3}
7	2.43	{1,2}
8	1.51	{2}
9	3.55	{1}
10	2.35	{2,3}

4.2   Weibull model

Consider again a system of m components and X_j denoting the failure time of the jth component, $j = 1, \ldots, m$. We assume that $X_1, X_2, \ldots, X_m$ are mutually independent. Let T the random variable that represents the system failure time and t a observation of T. Associated to each component j, let δ_j an indicator of censor.

The observation of X_j can be: $X_j = t$, the failure time of X_j it is not censored ($\delta_j = 1$); $X_j > t$, the failure time is right censored ($\delta_j = 2$); and $X_j \leq t$, the failure time is left censored ($\delta_j = 3$). Also, jth component can belong to the masked set or not.

Let s be the set of index indicating possible components that produced the failure of the system, that is, components that have their failure time masked, s is a subset of $\{1, \ldots, m\}$.

Let $t_1, \ldots, t_n$ a sample of system failure time of size n, and s_i is the set of masked components in the ith sample, $i = 1, \ldots, n$. Also, $v_{ji} = 1$ if the jth component has the failure time masked ($j \in s_i$ and s_i is not unitary),

and $v_{ji} = 0$ otherwise $(j \notin s_i)$, for $j = 1, \ldots, m$. The observation of jth component will be one of the following:

uncensored; not masked: $\delta_{ji} = 1$ and $v_{ji} = 0$;

right censored; not masked: $\delta_{ji} = 2$ and $v_{ji} = 0$;

left censored; not masked: $\delta_{ji} = 3$ and $v_{ji} = 0$;

masked: $v_{ji} = 1$.

If a component has the failure time masked, the component can be the one that produced the failure of the system (uncensored), right censored or left censored. Consider that

$$
\begin{aligned}
\lambda_{1j}(t) &= \Pr(v_j = 1 \mid t, \delta_j = 1), \\
\lambda_{2j}(t) &= \Pr(v_j = 1 \mid t, \delta_j = 2), \\
\lambda_{3j}(t) &= \Pr(v_j = 1 \mid t, \delta_j = 3),
\end{aligned}
$$

where $\lambda_{1j}(t)$ is the conditional probability of the jth component be masked, given the failure time of the system t, and the censor type $\delta_j = 1$. $\lambda_{2j}(t)$ and $\lambda_{3j}(t)$ are analogous to $\lambda_{1j}(t)$. Here, we consider that $\lambda_{1j}(t) = \lambda_{1j}$, $\lambda_{2j}(t) = \lambda_{2j}$, and $\lambda_{3j}(t) = \lambda_{3j}$, that is, the probability of a component be masked does not depend on the failure time t.

For each component, we can observe the triple $(t_i, \delta_{ji}, v_{ji})$, $i, \ldots, n$. Our interest consists in the estimation of the distribution function, F, of the j-h component. We consider a parametric family model for F with parameter $\boldsymbol{\theta}_j$, then the estimation of the parameter $\boldsymbol{\theta}_j$ induces the

distribution function F. The available information from the data is one of the following types:

1. $\Pr(X_{ji} \in (t_i, t_i], \upsilon_i = 0 \mid \boldsymbol{\theta}_j) = f(t_i \mid \boldsymbol{\theta}_j)(1 - \lambda_{1j})$, if the ith observation is uncensored and not masked;

2. $\Pr(X_{ji} \in (t_i, \infty), \upsilon_i = 0 \mid \boldsymbol{\theta}_j) = [1 - F(t_i \mid \boldsymbol{\theta}_j)](1 - \lambda_{2j})$, if the ith observation is right censored and not masked;

3. $\Pr(X_{ji} \in (0, t_i], \upsilon_i = 0 \mid \boldsymbol{\theta}_j) = F(t_i \mid \boldsymbol{\theta}_j)(1 - \lambda_{3j})$, if the ith observation is left censored and not masked;

4. $\Pr(X_{ji} \in (t_i, t_i], \upsilon_i = 1 \mid \boldsymbol{\theta}_j) = f(t_i \mid \boldsymbol{\theta}_j)\lambda_{1j}$, if the ith observation is uncensored and masked;

5. $\Pr(X_{ji} \in (t_i, \infty), \upsilon_i = 1 \mid \boldsymbol{\theta}_j) = [1 - F(t_i \mid \boldsymbol{\theta}_j)]\lambda_{2j}$, if the ith observation is right censored and masked; and

6. $\Pr(X_{ji} \in (0, t_i], \upsilon_i = 1 \mid \boldsymbol{\theta}_j) = F(t_i \mid \boldsymbol{\theta}_j)\lambda_{3j}$, if the ith observation is left censored and masked.

However, we do not have information about the cases 4 to 6, since when the data is masked, we do not know if that component was censored or not. Considering an augmented data procedure (latent variable), define $d_{1ji} = 1$ if the masked observation is not censored or $d_{1ji} = 0$ otherwise, $d_{2ji} = 1$ if the masked observation is right censored or $d_{2ji} = 0$ otherwise, and $d_{3ji} = 1$ if the masked observation is left censored or $d_{3ji} = 0$ otherwise. Besides, $\boldsymbol{d}_{ji} = (d_{1ji}, d_{2ji}, d_{3ji})$ and $\sum_{l=1}^{3} d_{lji} = 1$.

Let $R(t_i \mid \boldsymbol{\theta}_j) = 1 - F(t_i \mid \boldsymbol{\theta}_j)$ the reliability function. The likelihood function of the jth component can be writen as a part of non-masked

data and a part for masked data (augmented data), that is,

$$L(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j) = \prod_{i:\ v_{ji}=0} \left\{ [f(t_i \mid \boldsymbol{\theta}_j)\ (1 - \lambda_{1j})]^{I(\delta_{ji}=1)} \right.$$

$$\times [R(t_i \mid \boldsymbol{\theta}_j)\ (1 - \lambda_{2j})]^{I(\delta_{ji}=2)} [F(t_i \mid \boldsymbol{\theta}_j)\ (1 - \lambda_{3j})]^{I(\delta_{ji}=3)} \Big\}$$

$$\times \prod_{i:\ v_{ji}=1} \left\{ [f(t_i \mid \boldsymbol{\theta}_j)\ \lambda_{1j}]^{d_{1ji}} [R(t_i \mid \boldsymbol{\theta}_j)\ \lambda_{2j}]^{d_{2ji}} [F(t_i \mid \boldsymbol{\theta}_j)\ \lambda_{3j}]^{d_{3ji}} \right\} (4.1)$$

where $I(A) = 1$, if A is true, and $I(A) = 0$ otherwise, $\boldsymbol{t} = (t_1, \ldots, t_n)$, $\boldsymbol{v}_j = (v_{j1}, \ldots, v_{jn})$, $\boldsymbol{d}_j = (\boldsymbol{d}_{ji} : i \in \{v_{ji} = 1\})$ and $\boldsymbol{\delta}_j = (\delta_{ji} : i \in \{v_{ji} = 0\})$.

The likelihood function in (4.1) is defined generically and it is straightforward to any probability distribution. We propose the three-parameter Weibull distribution. Thus, the reliability and density functions are given in (3.1) and (3.2), respectively.

The estimation work is performed under a Bayesian perspective of inference, and thus, the priori distribution for $(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j)$ needs to be defined. The prior distributions of all parameters are considered independent with gamma distribution with mean 1 and variance 1000 for β_j, η_j, μ_j and uniform distribution over $(0, 1)$ for λ_{1j}, λ_{2j} and λ_{3j}. Besides, $\Pr(d_{lji} = 1) = \Pr(d_{lji} = 0) = 0.5$, for $l = 1, 2, 3$.

Here, no prior information about component's operation is known and noninformative prior is considered. However, it is possible to express a prior information about the component functioning in the system through the opinion of an expert and/or through past experiences.

The posterior density of $(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j)$ comes out to be

$$\pi(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j) \propto$$

$$\pi(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j) L(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j), \quad (4.2)$$

where $\pi(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j)$ is the prior distribution of $(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j)$.

The posterior density in Equation (4.2) has not close form. An alternative is to rely on Markov-Chain Monte-Carlo (MCMC) simulations. Here we consider Metropolis within Gibbs algorithm. This algorithm is suitable in this situation because it is possible direct sampling from conditional distribution for some parameters but for others this is not possible (Tierney, 1994). The algorithm works in the following steps:

1. *Attribute initial values* $\boldsymbol{\theta}_j^{(0)}$, $\lambda_{1j}^{(0)}$, $\lambda_{2j}^{(0)}$ *and* $\lambda_{3j}^{(0)}$ *for* $\boldsymbol{\theta}_j = (\beta_j, \eta_j, \mu_j)$, λ_{1j}, λ_{2j} *and* λ_{3j}, *respectively, and set* $b = 1$;

2. *For* $i \in \{\boldsymbol{v}_j = 1\}$, *draw* $\boldsymbol{d}_{ji}^{(b)}$ *from* $\pi(\boldsymbol{d}_{ji} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{\theta}_j^{(b-1)}, \lambda_{1j}^{(b-1)}, \lambda_{2j}^{(b-1)}, \lambda_{3j}^{(b-1)})$, *in which:*

$$\pi(\boldsymbol{d}_{ji} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}) \quad \propto \quad \left\{ \lambda_{1j} f(t_i|\boldsymbol{\theta}_j) \right\}^{d_{1ji}} \left\{ \lambda_{2j} R(t_i|\boldsymbol{\theta}_j) \right\}^{d_{2ji}}$$
$$\times \left\{ \lambda_{3j} F(t_i|\boldsymbol{\theta}_j) \right\}^{d_{3ji}},$$

that is, $\boldsymbol{d}_{ji} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j} \sim Multinomial(1; p_{1ji}, p_{2ji}, p_{3ji})$ *in which* $p_{1ji} = \lambda_{1j} f(t_i|\boldsymbol{\theta}_j)/C$, $p_{2ji} = \lambda_{2j} R(t_i|\boldsymbol{\theta}_j)/C$ *and* $p_{3ji} = \lambda_{3j} F(t_i|\boldsymbol{\theta}_j)/C$, *where* $C = \lambda_{1j} f(t_i|\boldsymbol{\theta}_j) + \lambda_{2j} R(t_i|\boldsymbol{\theta}_j) + \lambda_{3j} F(t_i|\boldsymbol{\theta}_j)$;

3. *Draw* $\boldsymbol{\theta}_j^{(b)}$ *from* $\pi(\boldsymbol{\theta}_j \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j^{(b)}, \lambda_{1j}^{(b-1)}, \lambda_{2j}^{(b-1)}, \lambda_{3j}^{(b-1)})$ *through Metropolis-Hastings algorithm (Robert e Casella, 2010), where*

$$\pi(\boldsymbol{\theta}_j \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}) \propto \quad \pi(\boldsymbol{\theta}_j) \prod_{i:\ v_{ji}=0} \left\{ [f(t_i \mid \boldsymbol{\theta}_j)\,(1 - \lambda_{1j})]^{I(\delta_{ji}=1)} \right.$$
$$\left. \times [R(t_i \mid \boldsymbol{\theta}_j)\,(1 - \lambda_{2j})]^{I(\delta_{ji}=2)} [F(t_i \mid \boldsymbol{\theta}_j)\,(1 - \lambda_{3j})]^{I(\delta_{ji}=3)} \right\}$$

$$\times \prod_{i:\ v_{ji}=1} \left\{ [f(t_i \mid \boldsymbol{\theta}_j)\ \lambda_{1j}]^{d_{1ji}} [R(t_i \mid \boldsymbol{\theta}_j)\ \lambda_{2j}]^{d_{2ji}} [F(t_i \mid \boldsymbol{\theta}_j)\ \lambda_{3j}]^{d_{3ji}} \right\}.$$

4. *Simulate λ_{1j} from $\pi(\lambda_{1j} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j^{(b)}, \boldsymbol{\theta}_j^{(b)}, \lambda_{2j}^{(b-1)}, \lambda_{3j}^{(b-1)})$ in which*

$$\pi(\lambda_{1j} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \boldsymbol{\theta}_j, \lambda_{2j}, \lambda_{3j}) \quad \propto \quad \lambda_{1j}^{\sum_{i:\ v_{ji}=1} d_{1ji}} (1 - \lambda_{1j})^{n_f},$$

that is, $\lambda_{1j} \mid (\boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \boldsymbol{\theta}_j, \lambda_{2j}, \lambda_{3j}) \sim Beta(\sum_{i:\ v_{ji}=1} d_{1ji}+1, n_f+1),$ in which n_f is the number of systems in which component j is known to be the responsible of system failure.

5. *Simulate λ_{2j} from $\pi(\lambda_{2j} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j^{(b)}, \boldsymbol{\theta}_j^{(b)}, \lambda_{1j}^{(b-1)}, \lambda_{3j}^{(b-1)})$ in which*

$$\pi(\lambda_{2j} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{3j}) \quad \propto \quad \lambda_{2j}^{\sum_{i:\ v_{ji}=1} d_{2ji}} (1 - \lambda_{2j})^{n_r},$$

that is, $\lambda_{2j} \mid (\boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{3j}) \sim Beta(\sum_{i:\ v_{ji}=1} d_{2ji} + 1, n_r + 1),$ in which n_r is the number of systems in which component j is observed to be right-censored.

6. *Simulate λ_{3j} from $\pi(\lambda_{3j} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j^{(b)}, \boldsymbol{\theta}_j^{(b)}, \lambda_{1j}^{(b-1)}, \lambda_{2j}^{(b-1)})$ in which*

$$\pi(\lambda_{3j} \mid \boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}) \quad \propto \quad \lambda_{3j}^{\sum_{i:\ v_{ji}=1} d_{3ji}} (1 - \lambda_{3j})^{n_l},$$

that is, $\lambda_{3j} \mid (\boldsymbol{t}, \boldsymbol{\delta}_j, \boldsymbol{v}_j, \boldsymbol{d}_j, \boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}) \sim Beta(\sum_{i:\ v_{ji}=1} d_{3ji} + 1, n_l + 1),$ in which n_l is the number of systems in which component j is observed to be left-censored.

7. *Let $b = b + 1$ and repeat steps 2) to 7) until $b = B$, where B is pre-set number of simulated samples of $(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j)$.*

Discarding burn-in (first generated values discarded to eliminate the effect of the assigned initial values for parameters) and jump samples (spacing among generated values to avoid correlation problems), a sample of size n_p from the joint posterior distribution of $(\boldsymbol{\theta}_j, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}, \boldsymbol{d}_j)$ is obtained. For the jth component, the sample from the posterior can be expressed as $(\boldsymbol{\theta}_j^{(1)}, \boldsymbol{\theta}_j^{(2)}, \ldots, \boldsymbol{\theta}_j^{(n_p)})$, $(\lambda_{1j}^{(1)}, \lambda_{1j}^{(2)}, \ldots, \lambda_{1j}^{(n_p)})$, $(\lambda_{2j}^{(1)}, \lambda_{2j}^{(2)}, \ldots, \lambda_{2j}^{(n_p)})$, $(\lambda_{3j}^{(1)}, \lambda_{3j}^{(2)}, \ldots, \lambda_{3j}^{(n_p)})$ and $(\boldsymbol{d}_j^{(1)}, \boldsymbol{d}_j^{(2)}, \ldots, \boldsymbol{d}_j^{(n_p)})$. Consequently, posterior quantities of reliability function $R(t \mid \boldsymbol{\theta}_j)$ can be easily obtained (Robert e Casella, 2010). For example, the posterior mean of the reliability function is given by

$$\mathrm{E}[R(t \mid \boldsymbol{\theta}_j) \mid Data] = \frac{1}{n_p} \sum_{k=1}^{n_p} R\left(t \mid \boldsymbol{\theta}_j^{(k)}\right), \quad \text{for each } t > 0.$$

4.3   Simulated datasets in masked scenario

We consider two simulated examples of complex system structure. The system 1, the 2-out-of-3 system is presented in Figure 1.7. System in Figure 4.1, system 2, can be denoted as a series system of components 1 and 2 with components 3, 4 and 5.
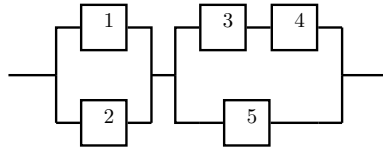
Figure 4.1: *System 2 representation.*

For these simulated examples, we consider that s_i consists of components that no longer work in the system failure, that is, there is no right censored for components in s_i, which leads to $\lambda_{2j} = 0$. Besides, for s_i

not unitary, $j \in s_i$ only if j belongs to the minimal cut set that caused the ith system failure. A cut set is a set of components which by failing causes the system to fail. A cut set is said to be minimal if it can not be reduced without loosing its status as a cut set. For the bridge system, for instance, there are four minimal cut sets, they are: $\{1,2\}$, $\{4,5\}$, $\{1,3,5\}$ and $\{2,3,4\}$.

Imagine a situation that this bridge system fail and the components 1, 2 and 3 do not work at the moment of system failure. Then, only components 1 and 2 belong to set s, once the component 3 does not belong to the minimal cut that caused the system failure and, in fact, the component 3 is observed to be left censored failure time. In this approach, we fit the proposed model under symmetric assumption (that is, $\lambda_{1j} = \lambda_{3j}$).

The simulated systems have the following characteristics:

- System 1 (2-out-of-3): $m = 3$ and X_1 was generated from a Weibull distribution with mean 15 and variance 8, X_2 from a gamma distribution with mean 18 and variance 12, X_3 from a lognormal distribution with mean 20 and variance 10 and the system failure time is $T = \max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\}$ Besides, $n = 300$ and $p = 0.4$, where p is the proportion of masked data system.

- System 2: $m = 5$ and X_1 was generated from a Weibull distribution with mean 12 and variance 15, X_2 from a gamma distribution with mean 11 and variance 11, X_3 from a three-parameter Weibull distribution with mean 12 and variance 9, X_4 from a log-

normal distribution with mean 12 and variance 7 and X_5 from a three-parameter Weibull distribution with mean 11 and variance 14. In this structure, the system lifetime is given by $T = \min\{\max\{X_1, X_2\}, \max\{\min\{X_3, X_4\}, X_5\}\}$, $n = 100$ and the proportion of masked data systems is $p = 0.3$.

For all systems, we generated 30,000 values of each parameter, disregarding the first 10,000 iterations as burn-in samples and jump of size 20 was chosen to avoid correlation problems, obtaining a sample of size $n_p = 1,000$. The chains convergence was monitored and good convergence results were obtained.

4.3.1   System 1

The generated data are presented in Table C.1 in Appendix C. Table 4.2 presents the percentages of observations types of each component. Component 1, for instance, has 46.67% of censored observations (40.67% to the left and 6% to the right) and it has 37.33% of systems in masked scenario.

Posterior measures of Weibull parameters β_j, η_j and μ_j, for $j = 1, 2, 3$, are presented in Table 4.3. The posterior quantities of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 4.4. The posterior means of reliability functions can be visualized in Figure 4.2 besides the empirical 95% HPD intervals. The reliability posterior means are close to the true reliability curves, in general. Even for values of t that the posterior mean is more distant from the true values, the lower or upper limit of HPD interval is very close to the true curve.

Table 4.2: *Percentages of type of observations for each component in 2-out-of-3 structure in masked scenario.*

Components	Type	Number of systems	%
1	Known failure	48	16.00%
	Left censored	122	40.67%
	Right censored	18	6.00%
	Masked	112	37.33%
2	Known failure	75	25.00%
	Left censored	49	16.33%
	Right censored	99	33.00%
	Masked	77	25.67%
3	Known failure	57	19.00%
	Left censored	9	3.00%
	Right censored	183	61.00%
	Masked	51	17.00%

Table 4.3: *Posterior measures of Weibull parameters for 2-out-of-3 components in masked scenario.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
				Componente 1				
β_1	3.722	4.894	5.242	5.250	5.645	7.090	0.532	4.209 - 6.244
η_1	14.420	15.440	15.650	15.630	15.850	16.610	0.305	15.029 - 16.207
μ_1	1.64E-09	4.16E-08	1.46E-07	4.42E-07	4.68E-07	3.30E-06	6.58E-07	1.63E-09 - 2.04E-06
				Componente 2				
	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
β_2	3.648	4.690	4.964	4.962	5.230	6.448	0.439	4.101 - 5.833
η_2	17.850	18.500	18.690	18.690	18.870	19.780	0.288	18.169 - 19.274
μ_2	1.08E-07	5.18E-07	9.55E-06	1.92E-05	2.70E-05	0.000221	3.05E-05	1.08E-07 - 6.13E-05
				Componente 3				
	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
β_3	5.308	6.695	7.118	7.143	7.557	9.216	0.610	6.008 - 8.296
η_3	19.640	20.380	20.620	20.610	20.820	21.830	0.339	19.987 - 21.333
μ_3	0.000	0.002	0.023	0.113	0.225	0.864	0.144	1.08E-04 - 0.362

Table 4.4: *Posterior measures of components' reliability functions involved in 2-out-of-3 masked system for some values of t.*

t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
				Component 1				
3.5	0.995	0.999	1.000	0.999	1.000	1.000	0.001	0.998 - 1.000
10.0	0.788	0.889	0.908	0.905	0.927	0.966	0.029	0.842 - 0.951
20.0	0.007	0.021	0.026	0.028	0.033	0.059	0.009	0.012 - 0.046
				Component 2				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
8.5	0.938	0.975	0.980	0.979	0.984	0.994	0.007	0.965 - 0.992
15.0	0.602	0.692	0.712	0.712	0.732	0.793	0.030	0.660 - 0.772
26.0	0.001	0.003	0.006	0.007	0.009	0.056	0.006	0.001 - 0.020
				Component 3				
t	Min	1Qt	Median	Mean	3Qt	Max	SD	CI 95%
12.5	0.936	0.968	0.974	0.973	0.978	0.989	0.008	0.958 - 0.986
20.0	0.347	0.437	0.460	0.458	0.480	0.555	0.034	0.392 - 0.528
25.0	0.001	0.012	0.021	0.024	0.033	0.119	0.016	0.002 - 0.056

(a) *Component 1*   (b) *Component 2*   (c) *Component 3*
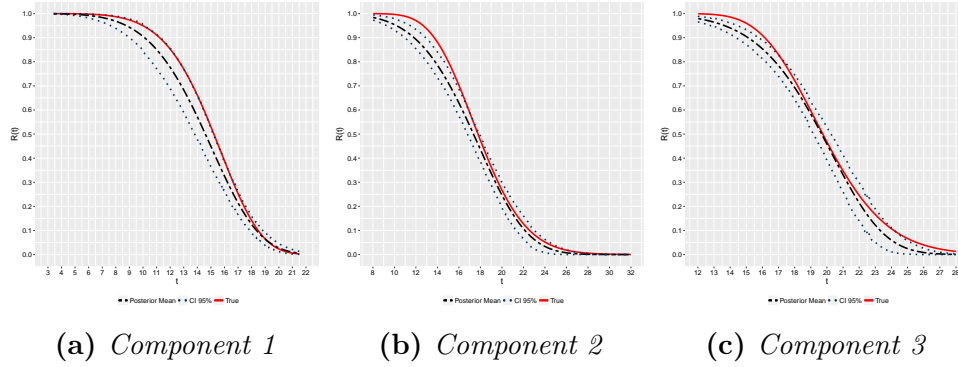
Figure 4.2: *Generating reliability functions, posterior means and 95% HPD intervals (CI 95%) for the components involved in 2-out-of-3 structure in masked scenario.*

4.3.2   System 2

The generated data are presented in Table C.5 in Appendix C. Table 4.5 presents the percentages of observations types of each component. Component 3, for instance, has 81% of censored observations (26% to the left and 55% to the right) and it has 9% of systems in masked scenario.

Posterior measures of Weibull parameters β_j, η_j and μ_j, for $j = 1, \ldots, 5$, are presented in Table 4.6. The posterior quantities of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 4.7. The posterior means of reliability functions can be visualized in Figure 4.3 besides the empirical 95% HPD intervals. The reliability posterior means are close to the true reliability curves, mainly for component 3. Even for values of t that the posterior mean is more distant from the true values, the lower or upper limit of HDP interval is very close to the true curve.

Table 4.5: *Percentages of type of observations for each component in 4.1 structure.*

Components	Type	Number of systems	%
1	Known failure	13	13.00%
	Left censored	37	37.00%
	Right censored	37	37.00%
	Masked	13	13.00%
2	Known failure	12	12.00%
	Left censored	48	48.00%
	Right censored	27	27.00%
	Masked	13	13.00%
3	Known failure	10	10.00%
	Left censored	26	26.00%
	Right censored	55	55.00%
	Masked	9	9.00%
4	Known failure	12	12.00%
	Left censored	30	30.00%
	Right censored	47	47.00%
	Masked	11	11.00%
5	Known failure	16	16.00%
	Left censored	54	54.00%
	Right censored	22	22.00%
	Masked	8	8.00%

Table 4.6: *Posterior measures of Weibull parameters for components in 4.1 structure in masked scenario.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
				Component 1				
β_1	1.017	2.188	2.568	2.572	2.922	4.449	0.558	1.561 - 3.677
η_1	10.650	12.190	12.700	12.780	13.290	16.590	0.815	11.287 - 14.242
μ_1	1.49E-33	6.03E-27	3.44E-24	8.25E-13	8.34E-19	2.48E-11	3.15E-12	0 - 0
				Component 2				
β_2	1.340	2.415	2.812	2.842	3.240	5.844	0.624	1.634 - 4.016
η_2	9.101	10.730	11.120	11.100	11.440	13.110	0.579	9.982 - 12.162
μ_2	9.79E-38	5.69E-35	2.86E-34	7.33E-23	6.66E-30	3.90E-21	4.50E-22	0 - 0
				Component 3				
β_3	2.283	4.195	4.789	4.844	5.454	8.149	0.944	3.130 - 6.838
η_3	11.690	12.660	12.960	13.040	13.360	15.530	0.566	12.023 - 14.212
μ_3	5.45E-19	1.49E-17	7.45E-16	7.03E-13	4.91E-13	1.52E-11	1.90E-12	0 - 0
				Component 4				
β_4	1.858	3.867	4.391	4.412	4.925	6.707	0.774	2.95 - 5.943
η_4	11.330	12.530	12.830	12.880	13.180	15.800	0.505	12.009 - 13.887
μ_4	4.09E-20	2.53E-18	2.13E-17	3.33E-12	1.20E-13	2.77E-10	2.40E-11	0 - 0
				Component 5				
β_5	1.559	2.562	2.914	2.932	3.300	4.524	0.534	1.947 - 3.96
η_5	9.526	10.820	11.120	11.130	11.450	13.190	0.513	10.224 - 12.307
μ_5	1.99E-23	2.11E-22	1.48E-19	4.95E-12	4.26E-16	1.92E-10	1.81E-11	0 - 0

Table 4.7: *Posterior measures of components reliability functions involved in structure 4.1 in masked scenario for some values of t.*

Component 1								
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
3.5	0.779	0.944	0.963	0.956	0.977	0.997	0.029	0.900 - 0.996
11.0	0.340	0.463	0.499	0.500	0.535	0.651	0.051	0.394 - 0.591
18.5	0.001	0.041	0.072	0.085	0.121	0.297	0.057	0.001 - 0.194
Component 2								
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
4.00	0.760	0.914	0.945	0.935	0.964	0.998	0.040	0.851 - 0.990
12.00	0.162	0.252	0.287	0.288	0.319	0.446	0.049	0.181 - 0.371
17.00	0.001	0.017	0.035	0.044	0.061	0.226	0.037	0.001 - 0.115
Component 3								
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
6.00	0.874	0.963	0.975	0.971	0.984	0.998	0.018	0.937 - 0.998
11.00	0.461	0.604	0.638	0.635	0.670	0.767	0.049	0.535 - 0.722
15.00	0.008	0.083	0.134	0.143	0.191	0.408	0.075	0.020 - 0.289
Component 4								
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
6.00	0.772	0.951	0.965	0.961	0.976	0.994	0.022	0.917 - 0.994
12.00	0.273	0.440	0.477	0.476	0.516	0.643	0.053	0.369 - 0.571
16.00	0.005	0.041	0.069	0.082	0.115	0.357	0.053	0.005 - 0.182
Component 5								
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC 95%
4.50	0.755	0.902	0.930	0.923	0.953	0.986	0.038	0.845 - 0.979
12.00	0.165	0.260	0.286	0.289	0.317	0.470	0.044	0.203 - 0.376
16.00	0.004	0.037	0.057	0.062	0.081	0.210	0.033	0.007 - 0.127

(a) *Component 1*   **(b)** *Component 2*   **(c)** *Component 3*

(d) *Component 4*   **(e)** *Component 5*
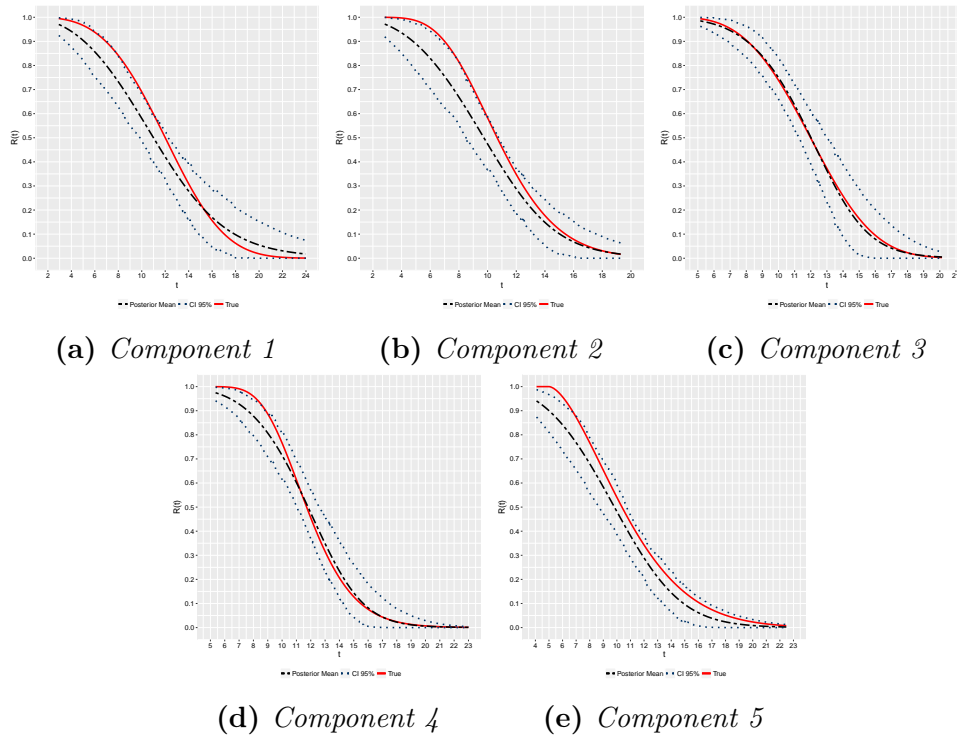
Figure 4.3: *Generating reliability functions, posterior means and 95% HPD intervals (CI 95%) for components involved in 4.1 structure in masked scenario.*

4.4   Computer hard-drive dataset

The dataset is available in Flehinger *et al.* (2002) which consists of 172 observed failure times of computer hard-drives monitored over a period of 4 years. There were three possible causes of failure: eletronic hard (component $j = 1$), head flyability (component $j = 2$) and head-/disc magnetics (component $j = 3$). That is, the hard-drives are system with three components in series. However, for some of them (38%) the cause of hard-drive fail was not identified. For these masked data systems, $s = \{1, 3\}$ or $s = \{1, 2, 3\}$, that is, there is no possible masked set $s = \{1, 2\}$ or $s = \{2, 3\}$. Note that in our proposal approach the configuration of set s is not a big deal, once the important information for estimation of jth component reliability is if j belongs to s or not. More details about the detection of failure causes can be found in Flehinger *et al.* (2001) and in Craiu e Reiser (2006). The dataset is presented in Table C.6 in Appendix C.

Since the components in s_i are right censored or responsible for system failure, $\lambda_{3j} = 0$, for $j = 1, 2, 3$, and no hard-drive is subject of left censored failure time. Table 4.8 presents the percentages of observations types of each component. The component 1 caused the failure of 20.35% of systems, 11.05% of systems failured because of component 2 and component 3 was responsible for 30.23% of system failure. Besides, or component 1 or component 3 caused the failure of 18.60% of systems and the remaining 19.77% of systems broken down because of any of three components.

We generated 35,000 values of each parameter, disregarding the first

5,000 iterations as burn-in samples and jump of size 30 to avoid correlation problems, obtaining a sample of size $n_p = 1,000$. The chains convergence was monitored and good convergence results were obtained.

Posterior measures of Weibull parameters β_j, η_j and μ_j, for $j = 1, 2, 3$, are presented in Table 4.9. The posterior quantities of $R(t \mid \boldsymbol{\theta}_j)$ for some values of t are shown in Table 4.10. The posterior means of reliability functions can be visualized in Figure 4.4 besides the empirical 95% HPD intervals.

Table 4.8: *Percentages of type of observations for each component in computer hard-drive dataset.*

Components	Type	Number of systems	%
1	Known failure	35	20.35%
	Right censored	71	41.28%
	Masked	66	38.37%
2	Known failure	19	11.05%
	Right censored	119	69.19%
	Masked	34	19.77%
3	Known failure	52	30.23%
	Right censored	54	31.40%
	Masked	66	38.37%

Table 4.9: *Posterior measures of Weibull parameters for components in computer hard drive dataset.*

	Min	1Qt	Median	Mean	3Qt	Max	SD	IC95%
				Component 1				
β_1	0.603	0.919	1.017	1.025	1.130	1.613	0.156	0.734 - 1.312
η_1	4.890	7.586	9.135	9.668	10.790	34.550	3.098	5.299 - 15.499
μ_1	2.03E-160	1.72E-129	3.55E-85	1.36E-35	1.03E-60	9.68E-33	3.11E-34	0 - 0
				Component 2				
β_2	0.607	1.295	1.480	1.502	1.684	2.584	0.305	0.994 - 2.184
η_2	5.168	8.151	9.783	10.870	12.060	76.190	5.027	5.350 - 19.275
μ_2	1.84E-201	2.45E-161	8.25E-120	2.17E-36	1.00E-78	7.55E-34	3.11E-35	0 - 0
				Component 3				
β_3	2.689	3.479	3.725	3.737	3.992	5.027	0.399	2.971 - 4.522
η_3	3.273	3.539	3.617	3.626	3.704	4.340	0.129	3.378 - 3.869
μ_3	4.67E-29	1.75E-21	1.93E-16	1.73E-11	8.58E-14	8.46E-10	7.75E-11	0 - 0

Table 4.10: *Posterior measures of components' reliability function involved in computer hard-drive dataset for some values of t.*

			Component 1					
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC95%
0.049	0.973	0.993	0.995	0.994	0.997	1.000	0.004	0.986 - 0.999
1.502	0.739	0.835	0.852	0.850	0.869	0.932	0.026	0.797 - 0.899
4.480	0.409	0.571	0.612	0.610	0.651	0.837	0.060	0.495 - 0.730
			Component 2					
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC95%
0.049	0.988	0.999	1.000	0.999	1.000	1.000	0.001	0.998 - 1.000
1.502	0.845	0.927	0.939	0.938	0.952	0.985	0.018	0.903 - 0.971
4.480	0.499	0.683	0.728	0.724	0.769	0.878	0.064	0.607 - 0.850
			Component 3					
t	Min	1Qt	Median	Mean	3Qt	Max	SD	IC95%
0.049	1.000	1.000	1.000	1.000	1.000	1.000	0.000	0.999 - 1.000
1.502	0.907	0.954	0.963	0.961	0.971	0.987	0.013	0.933 - 0.982
4.480	0.012	0.084	0.107	0.113	0.138	0.336	0.042	0.041 - 0.197
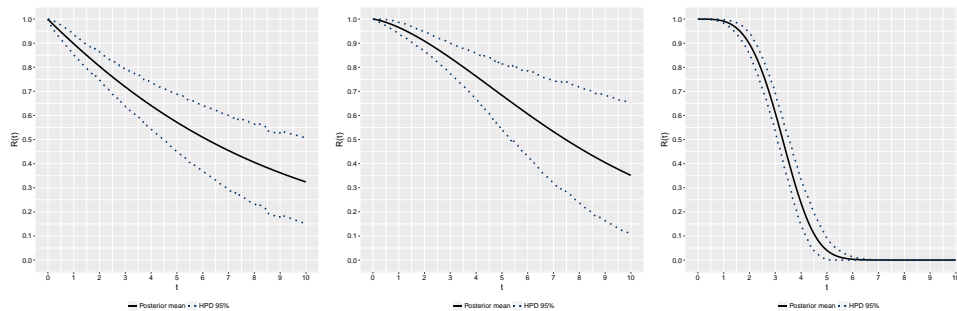
(a) *Eletronic hard*   **(b)** *Head flyability*   **(c)** *Head/disc magnetics*

Figure 4.4: *Generating reliability functions, posterior mean and 95% HPD intervals (CI 95%) for the components involved in computer hard-drive dataset.*

Appendix A

Datasets - Chapter 2

Table A.1: *Simulated sample of example 2.4.1*

T	δ	T	δ	T	δ	T	δ	T	δ
1.25	2	1.27	4	1.35	3	1.38	3	1.43	3
1.47	2	1.54	4	1.61	4	1.89	4	1.92	1
2.22	2	2.40	3	2.43	2	2.58	1	2.59	1
2.60	3	2.94	4	2.98	2	3.01	1	3.03	1
3.05	1	3.12	2	3.14	4	3.15	2	3.16	1
3.17	2	3.40	1	3.41	1	3.42	1	3.46	2
3.55	3	3.67	3	3.68	2	3.72	3	3.83	4
3.88	1	4.14	3	4.17	2	4.24	2	4.40	3
4.50	1	4.66	4	4.67	4	4.72	4	4.80	1
4.81	3	4.85	3	4.89	1	4.98	1	5.00	3
5.06	3	5.12	1	5.15	2	5.18	2	5.22	2
5.23	2	5.38	2	5.51	2	5.67	4	5.68	1
5.78	2	5.81	3	5.92	2	6.01	4	6.05	1
6.10	2	6.12	3	6.28	2	6.41	2	6.57	3
6.67	4	6.76	3	6.87	3	6.95	4	7.08	2
7.09	2	7.14	1	7.41	3	7.48	2	7.60	1
7.62	2	7.77	1	8.38	2	8.44	3	8.48	2
8.69	2	8.74	2	9.02	3	9.36	2	10.04	2
10.07	2	10.16	1	11.07	2	11.27	2	11.38	3
11.74	3	11.89	3	13.36	3	13.51	1	15.27	3

Table A.2: *Simulated sample of example 2.4.2*

T	δ	T	δ	T	δ	T	δ	T	δ
0.66	2	0.71	3	1.00	3	1.00	3	1.00	3
1.00	3	1.00	3	1.06	2	1.08	2	1.18	4
1.34	2	1.38	1	1.45	1	1.54	3	1.72	1
1.73	1	1.78	1	1.81	2	1.85	1	1.87	4
1.92	4	1.96	3	1.99	1	2.09	1	2.15	2
2.17	2	2.23	3	2.26	2	2.27	2	2.28	1
2.30	1	2.34	1	2.39	3	2.49	1	2.52	3
2.53	4	2.55	4	2.56	2	2.59	1	2.64	1
2.65	1	2.73	1	2.74	4	2.87	1	2.88	4
2.97	2	3.00	3	3.00	3	3.00	3	3.00	3
3.00	3	3.00	3	3.00	3	3.00	3	3.00	3
3.04	1	3.05	1	3.06	2	3.08	3	3.12	2
3.15	2	3.24	4	3.25	3	3.36	2	3.53	3
3.71	3	3.78	2	3.96	4	4.05	1	4.27	1
4.53	2	4.62	4	4.65	1	4.85	1	4.91	2
4.92	4	5.05	2	5.18	2	5.21	2	5.22	1
5.23	1	5.24	3	5.25	1	5.41	2	5.46	2
5.51	1	5.56	2	5.58	2	5.59	1	5.64	2
5.67	1	5.79	2	5.96	3	6.64	2	6.66	4
6.88	4	8.01	3	8.04	2	8.75	3	9.53	3

Appendix B

Datasets - Chapter 3

Table B.1: *Observed data of $n = 30$ parallel systems.*

System ID	Component 1		Component 2		Component 3	
	l	u	l	u	l	u
1	0.00	8.83	8.83	8.83	0.00	8.83
2	5.04	5.04	0.00	5.04	0.00	5.04
3	0.00	5.07	5.07	5.07	0.00	5.07
4	3.99	3.99	0.00	3.99	0.00	3.99
5	0.00	9.73	9.73	9.73	0.00	9.73
6	0.00	5.75	5.75	5.75	0.00	5.75
7	0.00	5.56	0.00	5.56	5.56	5.56
8	0.00	3.17	3.17	3.17	0.00	3.17
9	5.31	5.31	0.00	5.31	0.00	5.31
10	0.00	3.45	0.00	3.45	3.45	3.45
11	16.71	16.71	0.00	16.71	0.00	16.71
12	7.24	7.24	0.00	7.24	0.00	7.24
13	0.00	4.54	0.00	4.54	4.54	4.54
14	0.00	4.56	0.00	4.56	4.56	4.56
15	6.39	6.39	0.00	6.39	0.00	6.39
16	0.00	8.41	0.00	8.41	8.41	8.41
17	0.00	7.22	0.00	7.22	7.22	7.22
18	0.00	4.70	0.00	4.70	4.70	4.70
19	4.19	4.19	0.00	4.19	0.00	4.19
20	0.00	9.59	9.59	9.59	0.00	9.59
21	3.91	3.91	0.00	3.91	0.00	3.91
22	5.39	5.39	0.00	5.39	0.00	5.39
23	0.00	5.57	0.00	5.57	5.57	5.57
24	5.50	5.50	0.00	5.50	0.00	5.50
25	0.00	7.63	7.63	7.63	0.00	7.63
26	0.00	8.25	8.25	8.25	0.00	8.25
27	0.00	7.76	7.76	7.76	0.00	7.76
28	5.18	5.18	0.00	5.18	0.00	5.18
29	0.00	3.16	0.00	3.16	3.16	3.16
30	15.85	15.85	0.00	15.85	0.00	15.85

Table B.2: *Observed data of n = 50 2-out-of-3 systems.*

System ID	Component 1		Component 2		Component 3	
	l	u	l	u	l	u
1	0.00	20.16	20.16	20.16	20.16	∞
2	18.95	18.95	18.95	∞	0.00	18.95
3	0.00	20.12	20.12	20.12	20.12	∞
4	18.86	18.86	0.00	18.86	18.86	∞
5	0.00	15.80	15.80	15.80	15.80	∞
6	0.00	13.35	13.35	13.35	13.35	∞
7	0.00	14.32	14.32	∞	14.32	14.32
8	0.00	16.06	16.06	16.06	16.06	∞
9	0.00	19.65	19.65	19.65	19.65	∞
10	15.17	15.17	15.17	∞	0.00	15.17
11	0.00	19.36	19.36	19.36	19.36	∞
12	0.00	16.94	16.94	16.94	16.94	∞
13	0.00	16.09	16.09	16.09	16.09	∞
14	0.00	16.42	16.42	16.42	16.42	∞
15	17.06	∞	0.00	17.06	17.06	17.06
16	19.03	19.03	0.00	19.03	19.03	∞
17	0.00	17.11	17.11	17.11	17.11	∞
18	0.00	18.82	18.82	18.82	18.82	∞
19	0.00	16.63	16.63	∞	16.63	16.63
20	0.00	21.35	21.35	21.35	21.35	∞
21	16.63	16.63	16.63	∞	0.00	16.63
22	14.31	14.31	0.00	14.31	14.31	∞
23	0.00	15.90	15.90	15.90	15.90	∞
24	0.00	16.09	16.09	∞	16.09	16.09
25	0.00	15.74	15.74	15.74	15.74	∞
26	0.00	15.60	15.60	∞	15.60	15.60
27	0.00	18.55	18.55	18.55	18.55	∞
28	0.00	17.11	17.11	17.11	17.11	∞
29	0.00	22.14	22.14	∞	22.14	22.14
30	0.00	17.86	17.86	∞	17.86	17.86
31	0.00	16.45	16.45	16.45	16.45	∞
32	0.00	17.24	17.24	17.24	17.24	∞
33	14.93	14.93	0.00	14.93	14.93	∞
34	15.68	15.68	0.00	15.68	15.68	∞
35	0.00	17.30	17.30	17.30	17.30	∞
36	0.00	18.58	18.58	∞	18.58	18.58
37	0.00	18.44	18.44	∞	18.44	18.44
38	0.00	17.27	17.27	17.27	17.27	∞
39	0.00	18.33	18.33	18.33	18.33	∞
40	20.47	20.47	0.00	20.47	20.47	∞
41	0.00	16.60	16.60	16.60	16.60	∞
42	0.00	20.67	20.67	∞	20.67	20.67
43	17.34	17.34	17.34	∞	0.00	17.34
44	0.00	16.57	16.57	16.57	16.57	∞
45	16.11	16.11	0.00	16.11	16.11	∞
46	0.00	20.49	20.49	20.49	20.49	∞
47	15.28	15.28	0.00	15.28	15.28	∞
48	0.00	15.94	15.94	15.94	15.94	∞
49	19.72	∞	0.00	19.72	19.72	19.72
50	15.64	∞	0.00	15.64	15.64	15.64

Table B.3: *Observed data of n = 100 bridge systems (continues on the next page).*

System ID	Component 1		Component 2		Component 3		Component 4		Component 5	
	l	u	l	u	l	u	l	u	l	u
1	16.56	∞	0.00	16.56	0.00	16.56	16.56	16.56	16.56	∞
2	18.98	18.98	0.00	18.98	18.98	∞	0.00	18.98	18.98	∞
3	0.00	17.50	17.50	17.50	0.00	17.50	0.00	17.50	17.50	∞
4	15.21	∞	15.21	∞	0.00	15.21	15.21	15.21	0.00	15.21
5	0.00	19.49	19.49	19.49	0.00	19.49	19.49	∞	19.49	∞
6	16.37	∞	0.00	16.37	16.37	16.37	0.00	16.37	16.37	∞
7	0.00	17.33	17.33	∞	0.00	17.33	17.33	∞	17.33	17.33
8	0.00	14.48	14.48	14.48	14.48	∞	0.00	14.48	14.48	∞
9	19.54	∞	0.00	19.54	19.54	∞	0.00	19.54	19.54	19.54
10	17.69	17.69	0.00	17.69	0.00	17.69	17.69	∞	0.00	17.69
11	16.40	∞	16.40	∞	0.00	16.40	0.00	16.40	16.40	16.40
12	17.22	∞	0.00	17.22	0.00	17.22	17.22	17.22	17.22	∞
13	0.00	21.87	21.87	21.87	0.00	21.87	0.00	21.87	21.87	∞
14	16.53	∞	16.53	∞	0.00	16.53	0.00	16.53	16.53	16.53
15	15.48	∞	0.00	15.48	15.48	15.48	0.00	15.48	15.48	∞
16	14.08	14.08	0.00	14.08	14.08	∞	0.00	14.08	14.08	∞
17	0.00	13.84	13.84	13.84	13.84	∞	13.84	∞	13.84	∞
18	14.27	∞	0.00	14.27	14.27	14.27	0.00	14.27	14.27	∞
19	0.00	17.52	17.52	17.52	0.00	17.52	17.52	∞	17.52	∞
20	0.00	18.38	18.38	∞	0.00	18.38	0.00	18.38	18.38	18.38
21	14.50	∞	0.00	14.50	14.50	14.50	0.00	14.50	14.50	∞
22	15.63	15.63	0.00	15.63	15.63	∞	0.00	15.63	15.63	∞
23	0.00	13.83	13.83	13.83	0.00	13.83	13.83	∞	13.83	∞
24	0.00	14.86	14.86	∞	0.00	14.86	14.86	∞	14.86	14.86
25	16.77	∞	0.00	16.77	16.77	16.77	0.00	16.77	16.77	∞
26	0.00	19.55	19.55	19.55	0.00	19.55	0.00	19.55	19.55	∞
27	0.00	19.27	19.27	19.27	0.00	19.27	0.00	19.27	19.27	∞
28	16.73	16.73	0.00	16.73	0.00	16.73	16.73	16.73	16.73	∞
29	16.98	∞	16.98	∞	0.00	16.98	0.00	16.98	16.98	16.98
30	13.21	∞	0.00	13.21	13.21	13.21	0.00	13.21	13.21	∞
31	0.00	15.79	15.79	15.79	15.79	∞	0.00	15.79	15.79	∞
32	15.79	∞	0.00	15.79	15.79	∞	15.79	15.79	0.00	15.79
33	18.68	∞	18.68	∞	0.00	18.68	0.00	18.68	18.68	18.68
34	0.00	12.55	12.55	12.55	12.55	∞	12.55	∞	12.55	∞
35	15.69	∞	0.00	15.69	0.00	15.69	15.69	15.69	15.69	∞
36	0.00	19.26	19.26	19.26	0.00	19.26	0.00	19.26	19.26	∞
37	0.00	23.57	23.57	23.57	0.00	23.57	0.00	23.57	23.57	∞
38	15.78	15.78	0.00	15.78	15.78	∞	0.00	15.78	15.78	∞
39	18.31	18.31	0.00	18.31	0.00	18.31	18.31	∞	0.00	18.31
40	16.00	16.00	0.00	16.00	16.00	∞	0.00	16.00	16.00	∞
41	17.42	∞	17.42	17.42	0.00	17.42	0.00	17.42	17.42	∞
42	13.23	∞	0.00	13.23	0.00	13.23	13.23	13.23	13.23	∞
43	13.29	∞	0.00	13.29	13.29	∞	13.29	13.29	0.00	13.29
44	17.19	∞	0.00	17.19	0.00	17.19	17.19	17.19	17.19	∞
45	0.00	13.68	13.68	∞	13.68	∞	13.68	13.68	0.00	13.68
46	0.00	13.80	13.80	13.80	0.00	13.80	13.80	∞	13.80	∞
47	13.57	∞	13.57	∞	0.00	13.57	0.00	13.57	13.57	13.57
48	16.04	∞	0.00	16.04	16.04	16.04	0.00	16.04	16.04	∞
49	16.93	∞	16.93	∞	0.00	16.93	16.93	16.93	0.00	16.93
50	0.00	17.55	17.55	17.55	0.00	17.55	0.00	17.55	17.55	∞

Table B.4: *Continuation - observed data of n = 100 bridge systems.*

System ID	Component 1		Component 2		Component 3		Component 4		Component 5	
	l	u	l	u	l	u	l	u	l	u
51	18.12	∞	18.12	18.12	0.00	18.12	0.00	18.12	18.12	∞
52	17.66	∞	0.00	17.66	17.66	17.66	0.00	17.66	17.66	∞
53	14.93	14.93	0.00	14.93	0.00	14.93	14.93	∞	14.93	∞
54	19.76	∞	0.00	19.76	0.00	19.76	19.76	19.76	19.76	∞
55	0.00	21.56	21.56	∞	21.56	∞	0.00	21.56	21.56	21.56
56	13.89	∞	0.00	13.89	13.89	13.89	0.00	13.89	13.89	∞
57	16.93	∞	16.93	∞	0.00	16.93	0.00	16.93	16.93	16.93
58	0.00	17.62	17.62	∞	17.62	∞	17.62	17.62	0.00	17.62
59	0.00	21.44	21.44	21.44	0.00	21.44	0.00	21.44	21.44	∞
60	14.11	∞	14.11	14.11	0.00	14.11	0.00	14.11	14.11	∞
61	13.65	∞	13.65	∞	0.00	13.65	13.65	13.65	0.00	13.65
62	12.93	∞	0.00	12.93	0.00	12.93	12.93	12.93	12.93	∞
63	18.67	18.67	0.00	18.67	18.67	∞	0.00	18.67	18.67	∞
64	14.86	∞	14.86	14.86	0.00	14.86	0.00	14.86	14.86	∞
65	17.47	∞	17.47	∞	0.00	17.47	0.00	17.47	17.47	17.47
66	0.00	15.57	15.57	∞	0.00	15.57	15.57	∞	15.57	15.57
67	17.17	∞	0.00	17.17	0.00	17.17	17.17	17.17	17.17	∞
68	16.90	∞	0.00	16.90	0.00	16.90	16.90	16.90	16.90	∞
69	13.56	∞	0.00	13.56	0.00	13.56	13.56	13.56	13.56	∞
70	16.82	∞	16.82	∞	0.00	16.82	0.00	16.82	16.82	16.82
71	14.96	∞	0.00	14.96	14.96	∞	0.00	14.96	14.96	14.96
72	16.98	16.98	16.98	∞	0.00	16.98	16.98	∞	0.00	16.98
73	14.89	∞	14.89	14.89	0.00	14.89	0.00	14.89	14.89	∞
74	0.00	19.49	19.49	∞	0.00	19.49	0.00	19.49	19.49	19.49
75	0.00	15.17	15.17	15.17	15.17	∞	15.17	∞	15.17	∞
76	14.83	∞	0.00	14.83	0.00	14.83	14.83	14.83	14.83	∞
77	14.51	14.51	0.00	14.51	14.51	∞	0.00	14.51	14.51	∞
78	14.92	∞	0.00	14.92	0.00	14.92	14.92	14.92	14.92	∞
79	16.97	∞	0.00	16.97	16.97	16.97	0.00	16.97	16.97	∞
80	0.00	15.53	15.53	15.53	0.00	15.53	0.00	15.53	15.53	∞
81	17.73	17.73	0.00	17.73	0.00	17.73	17.73	∞	17.73	∞
82	14.91	∞	14.91	∞	14.91	∞	14.91	14.91	0.00	14.91
83	0.00	16.80	16.80	16.80	16.80	∞	0.00	16.80	16.80	∞
84	16.12	∞	0.00	16.12	0.00	16.12	16.12	16.12	16.12	∞
85	16.41	∞	16.41	16.41	0.00	16.41	0.00	16.41	16.41	∞
86	13.57	∞	0.00	13.57	0.00	13.57	13.57	13.57	13.57	∞
87	16.28	∞	0.00	16.28	0.00	16.28	16.28	16.28	16.28	∞
88	16.15	∞	0.00	16.15	0.00	16.15	16.15	16.15	16.15	∞
89	16.46	16.46	0.00	16.46	16.46	∞	0.00	16.46	16.46	∞
90	21.26	21.26	0.00	21.26	21.26	∞	0.00	21.26	21.26	∞
91	15.34	∞	0.00	15.34	0.00	15.34	15.34	15.34	15.34	∞
92	10.90	10.90	0.00	10.90	10.90	∞	10.90	10.90	10.90	∞
93	0.00	19.03	19.03	19.03	0.00	19.03	0.00	19.03	19.03	∞
94	0.00	17.06	17.06	∞	0.00	17.06	0.00	17.06	17.06	17.06
95	16.40	16.40	0.00	16.40	0.00	16.40	16.40	∞	16.40	∞
96	0.00	14.63	14.63	14.63	0.00	14.63	0.00	14.63	14.63	∞
97	19.26	∞	0.00	19.26	19.26	19.26	0.00	19.26	19.26	∞
98	0.00	17.60	17.60	17.60	17.60	∞	0.00	17.60	17.60	∞
99	19.22	∞	19.22	19.22	0.00	19.22	0.00	19.22	19.22	∞
100	16.51	∞	0.00	16.51	0.00	16.51	16.51	16.51	16.51	∞

Table B.5: *Device-G observed data.*

Device-G ID	Component W		Component S	
	l	u	l	u
1	275	275	275	∞
2	13	∞	13	13
3	147	147	147	∞
4	23	∞	23	23
5	181	181	181	∞
6	30	∞	30	30
7	65	∞	65	65
8	10	∞	10	10
9	300	∞	300	∞
10	173	∞	173	173
11	106	∞	106	106
12	300	∞	300	∞
13	300	∞	300	∞
14	212	212	212	∞
15	300	∞	300	∞
16	300	∞	300	∞
17	300	∞	300	∞
18	2	∞	2	2
19	261	∞	261	261
20	293	293	293	∞
21	88	∞	88	88
22	247	∞	247	247
23	28	∞	28	28
24	143	∞	143	143
25	300	∞	300	∞
26	23	∞	23	23
27	300	∞	300	∞
28	80	∞	80	80
29	245	245	245	∞
30	266	266	266	∞

Appendix C

Datasets - Chapter 4

Table C.1: *Observed data of $n = 300$ 2-out-of-3 systems in masked scenario (continues on the next pages).*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3
1	16.91	3	0	1	0	2	0
2	19.31	2	0	3	0	1	0
3	17.12	3	0	2	0	1	0
4	19.17	-	1	2	0	-	1
5	15.58	3	0	1	0	2	0
6	18.04	3	0	1	0	2	0
7	17.89	1	0	3	0	2	0
8	15.39	1	0	3	0	2	0
9	12.71	-	1	-	1	2	0
10	19.82	1	0	3	0	2	0
11	19.49	1	0	3	0	2	0
12	16.38	1	0	2	0	3	0
13	23.86	3	0	1	0	2	0
14	15.74	1	0	3	0	2	0
15	18.55	3	0	2	0	1	0
16	15.32	3	0	1	0	2	0
17	14.69	1	0	3	0	2	0
18	16.85	3	0	2	0	1	0
19	19.62	1	0	3	0	2	0
20	19.41	1	0	3	0	2	0
21	18.88	3	0	2	0	1	0
22	17.67	-	1	-	1	2	0
23	16.09	3	0	1	0	2	0
24	23.96	3	0	1	0	2	0
25	15.52	3	0	1	0	2	0
26	18.82	1	0	3	0	2	0
27	15.06	1	0	3	0	2	0
28	16.42	3	0	2	0	1	0
29	16.77	-	1	-	1	2	0
30	17.21	3	0	1	0	2	0
31	16.70	1	0	3	0	2	0
32	20.66	-	1	2	0	-	1
33	19.46	3	0	2	0	1	0
34	13.22	-	1	-	1	2	0
35	18.83	-	1	-	1	2	0
36	16.91	-	1	2	0	-	1
37	18.23	-	1	2	0	-	1
38	17.14	-	1	2	0	-	1
39	14.45	3	0	2	0	1	0
40	18.57	1	0	3	0	2	0

Table C.2: *Continuation of Table C.1 (continues on the next pages).*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3
41	19.87	-	1	2	0	-	1
42	15.52	3	0	1	0	2	0
43	16.28	3	0	1	0	2	0
44	19.72	3	0	2	0	1	0
45	18.44	3	0	1	0	2	0
46	19.92	3	0	2	0	1	0
47	17.24	1	0	3	0	2	0
48	13.34	-	1	-	1	2	0
49	14.51	-	1	-	1	2	0
50	14.91	1	0	3	0	2	0
51	15.97	1	0	3	0	2	0
52	15.79	-	1	-	1	2	0
53	15.33	3	0	1	0	2	0
54	18.07	1	0	3	0	2	0
55	17.46	3	0	2	0	1	0
56	19.11	-	1	2	0	-	1
57	17.86	3	0	1	0	2	0
58	17.46	3	0	2	0	1	0
59	20.46	3	0	2	0	1	0
60	15.73	-	1	-	1	2	0
61	19.31	1	0	3	0	2	0
62	16.91	1	0	2	0	3	0
63	17.22	2	0	3	0	1	0
64	16.70	-	1	-	1	2	0
65	19.33	-	1	-	1	2	0
66	16.25	3	0	1	0	2	0
67	17.12	3	0	1	0	2	0
68	19.38	-	1	2	0	-	1
69	16.53	3	0	1	0	2	0
70	16.13	3	0	1	0	2	0
71	17.45	2	0	1	0	3	0
72	19.09	3	0	1	0	2	0
73	20.38	3	0	2	0	1	0
74	17.92	3	0	2	0	1	0
75	19.51	-	1	-	1	2	0
76	14.42	-	1	-	1	2	0
77	18.51	2	0	-	1	-	1
78	18.97	-	1	-	1	2	0
79	17.94	-	1	2	0	-	1
80	18.84	2	0	3	0	1	0
81	14.92	1	0	3	0	2	0
82	22.99	-	1	2	0	-	1
83	15.07	2	0	1	0	3	0
84	17.71	3	0	2	0	1	0
85	17.72	1	0	2	0	3	0
86	20.89	3	0	1	0	2	0
87	16.72	3	0	1	0	2	0
88	15.81	-	1	-	1	2	0
89	15.12	-	1	2	0	-	1
90	15.74	-	1	-	1	2	0
91	19.85	3	0	2	0	1	0
92	18.10	-	1	2	0	-	1
93	17.77	-	1	-	1	2	0
94	18.59	-	1	-	1	2	0
95	16.39	-	1	-	1	2	0
96	17.43	3	0	2	0	1	0
97	19.85	3	0	1	0	2	0
98	17.21	2	0	-	1	-	1
99	14.93	-	1	-	1	2	0
100	16.78	3	0	1	0	2	0
101	16.16	1	0	2	0	3	0
102	18.70	3	0	1	0	2	0
103	19.52	3	0	2	0	1	0
104	20.51	3	0	2	0	1	0
105	17.58	2	0	1	0	3	0
106	15.85	1	0	3	0	2	0
107	16.89	3	0	1	0	2	0
108	18.40	3	0	2	0	1	0
109	17.15	-	1	2	0	-	1
110	15.48	-	1	2	0	-	1
111	19.61	-	1	2	0	-	1
112	14.99	-	1	2	0	-	1
113	18.56	3	0	2	0	1	0
114	15.00	-	1	2	0	-	1
115	18.12	2	0	-	1	-	1
116	18.27	-	1	2	0	-	1
117	17.08	3	0	1	0	2	0
118	16.24	3	0	1	0	2	0
119	17.27	3	0	1	0	2	0
120	15.54	3	0	2	0	1	0
121	20.99	3	0	1	0	2	0
122	21.00	-	1	-	1	2	0
123	17.25	3	0	2	0	1	0
124	14.92	-	1	-	1	2	0
125	21.30	-	1	-	1	2	0
126	13.24	-	1	2	0	-	1
127	18.60	3	0	1	0	2	0
128	19.34	3	0	1	0	2	0
129	19.11	1	0	3	0	2	0
130	15.65	1	0	3	0	2	0
131	17.15	3	0	2	0	1	0
132	17.21	1	0	3	0	2	0
133	17.02	3	0	2	0	1	0
134	16.77	3	0	1	0	2	0
135	14.85	3	0	1	0	2	0

Table C.3: *Continuation of Table C.1 (continues on the next pages).*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3
136	16.61	3	0	1	0	2	0
137	21.86	3	0	2	0	1	0
138	18.65	3	0	1	0	2	0
139	17.09	-	1	-	1	2	0
140	17.23	3	0	2	0	1	0
141	18.78	-	1	-	1	2	0
142	19.94	-	1	2	0	-	1
143	15.79	-	1	2	0	-	1
144	16.16	-	1	2	0	-	1
145	18.60	3	0	2	0	1	0
146	17.27	1	0	3	0	2	0
147	18.63	3	0	1	0	2	0
148	15.88	-	1	-	1	2	0
149	17.37	-	1	-	1	2	0
150	16.48	3	0	1	0	2	0
151	14.61	3	0	1	0	2	0
152	15.69	-	1	2	0	-	1
153	15.79	-	1	2	0	-	1
154	16.66	3	0	2	0	1	0
155	20.69	2	0	3	0	1	0
156	13.89	-	1	2	0	-	1
157	20.72	-	1	2	0	-	1
158	14.09	-	1	-	1	2	0
159	15.09	-	1	-	1	2	0
160	18.32	-	1	2	0	-	1
161	19.23	3	0	1	0	2	0
162	16.88	3	0	2	0	1	0
163	17.26	3	0	2	0	1	0
164	20.15	1	0	3	0	2	0
165	15.07	-	1	-	1	2	0
166	19.37	3	0	1	0	2	0
167	14.74	1	0	3	0	2	0
168	17.58	3	0	1	0	2	0
169	18.53	-	1	2	0	-	1
170	19.87	3	0	1	0	2	0
171	17.32	3	0	2	0	1	0
172	23.58	3	0	1	0	2	0
173	16.80	-	1	-	1	2	0
174	16.13	1	0	3	0	2	0
175	17.41	1	0	3	0	2	0
176	18.18	-	1	-	1	2	0
177	14.88	3	0	1	0	2	0
178	16.17	-	1	-	1	2	0
179	12.14	3	0	1	0	2	0
180	18.35	1	0	3	0	2	0
181	20.28	1	0	3	0	2	0
182	16.74	3	0	1	0	2	0
183	17.61	-	1	2	0	-	1
184	17.45	-	1	2	0	-	1
185	17.33	3	0	2	0	1	0
186	16.65	3	0	1	0	2	0
187	17.47	3	0	2	0	1	0
188	20.64	3	0	1	0	2	0
189	17.16	-	1	-	1	2	0
190	20.18	3	0	1	0	2	0
191	18.13	2	0	3	0	1	0
192	14.92	1	0	3	0	2	0
193	19.13	3	0	1	0	2	0
194	15.50	3	0	2	0	1	0
195	17.44	-	1	2	0	-	1
196	19.16	-	1	-	1	2	0
197	17.27	3	0	1	0	2	0
198	12.25	3	0	1	0	2	0
199	21.15	3	0	2	0	1	0
200	18.96	-	1	-	1	2	0
201	18.02	-	1	-	1	2	0
202	18.06	3	0	2	0	1	0
203	18.37	2	0	-	1	-	1
204	19.00	3	0	2	0	1	0
205	16.61	1	0	3	0	2	0
206	16.04	2	0	-	1	-	1
207	19.09	-	1	-	1	2	0
208	19.79	3	0	2	0	1	0
209	18.33	-	1	-	1	2	0
210	16.67	2	0	1	0	3	0
211	18.89	-	1	-	1	2	0
212	18.02	3	0	2	0	1	0
213	21.58	-	1	2	0	-	1
214	15.38	1	0	3	0	2	0
215	16.16	3	0	2	0	1	0
216	19.15	1	0	3	0	2	0
217	18.50	1	0	2	0	3	0
218	13.89	1	0	3	0	2	0
219	17.63	3	0	1	0	2	0
220	19.56	-	1	-	1	2	0
221	15.24	2	0	-	1	-	1
222	17.91	-	1	-	1	2	0
223	14.81	-	1	-	1	2	0
224	17.64	3	0	1	0	2	0
225	17.27	-	1	-	1	2	0
226	16.02	-	1	-	1	2	0
227	20.84	3	0	2	0	1	0
228	17.19	3	0	1	0	2	0
229	18.30	1	0	3	0	2	0
230	18.40	1	0	3	0	2	0

Table C.4: *Continuation of Table C.1.*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3
231	15.69	-	1	-	1	2	0
232	15.98	3	0	2	0	1	0
233	17.80	-	1	2	0	-	1
234	16.55	3	0	2	0	1	0
235	20.64	-	1	2	0	-	1
236	19.07	3	0	1	0	2	0
237	16.41	-	1	-	1	2	0
238	17.51	-	1	-	1	2	0
239	15.20	3	0	2	0	1	0
240	16.37	3	0	1	0	2	0
241	12.10	3	0	1	0	2	0
242	21.94	3	0	1	0	2	0
243	18.64	-	1	-	1	2	0
244	14.55	3	0	1	0	2	0
245	18.32	-	1	-	1	2	0
246	22.90	-	1	-	1	2	0
247	16.40	3	0	2	0	1	0
248	16.62	2	0	-	1	-	1
249	16.69	-	1	2	0	-	1
250	16.13	3	0	1	0	2	0
251	17.64	-	1	2	0	-	1
252	14.90	-	1	-	1	2	0
253	21.53	-	1	2	0	-	1
254	16.88	-	1	-	1	2	0
255	16.99	1	0	3	0	2	0
256	17.45	-	1	-	1	2	0
257	16.46	3	0	1	0	2	0
258	17.02	2	0	-	1	-	1
259	14.39	3	0	1	0	2	0
260	16.80	-	1	-	1	2	0
261	17.04	1	0	3	0	2	0
262	13.86	-	1	-	1	2	0
263	18.99	3	0	1	0	2	0
264	15.30	-	1	2	0	-	1
265	15.25	1	0	3	0	2	0
266	17.67	3	0	2	0	1	0
267	19.95	-	1	-	1	2	0
268	17.91	3	0	1	0	2	0
269	19.34	3	0	2	0	1	0
270	20.29	-	1	2	0	-	1
271	15.47	1	0	3	0	2	0
272	19.85	2	0	3	0	1	0
273	16.33	1	0	3	0	2	0
274	21.96	-	1	2	0	-	1
275	17.49	-	1	-	1	2	0
276	18.01	3	0	1	0	2	0
277	15.25	3	0	1	0	2	0
278	14.77	-	1	-	1	2	0
279	18.61	3	0	1	0	2	0
280	15.35	-	1	-	1	2	0
281	16.77	-	1	-	1	2	0
282	20.02	3	0	1	0	2	0
283	16.98	3	0	1	0	2	0
284	17.00	-	1	2	0	-	1
285	21.59	3	0	2	0	1	0
286	20.09	3	0	2	0	1	0
287	18.47	1	0	3	0	2	0
288	14.25	3	0	1	0	2	0
289	20.32	-	1	-	1	2	0
290	21.25	-	1	-	1	2	0
291	20.21	-	1	-	1	2	0
292	20.80	3	0	2	0	1	0
293	13.34	-	1	-	1	2	0
294	19.98	-	1	-	1	2	0
295	16.87	-	1	2	0	-	1
296	15.78	-	1	2	0	-	1
297	24.28	-	1	-	1	2	0
298	20.09	-	1	-	1	2	0
299	19.03	-	1	-	1	2	0
300	18.48	1	0	3	0	2	0

Table C.5: *Observed data of n = 100 systems as 4.1 in masked scenario.*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3	δ_4	v_4	δ_5	v_5
1	11.39	3	0	1	0	2	0	2	0	2	0
2	10.12	2	0	3	0	2	0	3	0	3	0
3	11.87	2	0	2	0	1	0	3	0	3	0
4	11.64	1	0	3	0	-	1	-	1	-	1
5	10.01	3	0	2	0	3	0	-	1	1	1
6	11.25	3	0	2	0	2	0	3	0	3	0
7	9.30	3	0	3	0	2	0	2	0	2	0
8	9.56	3	0	1	0	2	0	-	1	-	1
9	12.99	1	0	3	0	3	0	3	0	2	0
10	11.43	1	0	3	0	2	0	2	0	3	0
11	13.84	2	0	3	0	2	0	3	0	3	0
12	9.59	-	1	-	1	3	0	2	0	3	0
13	10.24	-	1	-	1	3	0	2	0	3	0
14	9.20	2	0	2	0	2	0	3	0	1	0
15	10.44	2	0	3	0	2	0	1	0	3	0
16	8.92	2	0	3	0	2	0	3	0	1	0
17	14.41	2	0	2	0	1	0	2	0	3	0
18	8.75	-	1	-	1	2	0	2	0	3	0
19	11.10	3	0	1	0	2	0	3	0	2	0
20	9.11	2	0	3	0	2	0	1	0	3	0
21	8.92	1	0	3	0	2	0	2	0	3	0
22	14.93	2	0	3	0	-	1	2	0	3	1
23	8.09	-	1	-	1	2	0	3	0	3	0
24	11.14	3	0	2	0	3	0	2	0	1	0
25	14.22	3	0	3	0	2	0	2	0	2	0
26	12.56	2	0	2	0	1	0	2	0	3	0
27	14.31	3	0	2	0	2	0	1	0	3	0
28	9.76	-	1	-	1	2	0	1	0	3	0
29	11.44	2	0	2	0	1	0	2	0	3	0
30	12.71	3	0	3	0	2	0	2	0	3	0
31	8.78	2	0	2	0	3	0	2	0	3	0
32	12.28	3	0	2	0	2	0	1	0	3	0
33	8.72	2	0	2	0	1	0	2	0	3	0
34	9.32	2	0	3	0	2	0	1	0	3	0
35	8.09	3	0	3	0	2	0	2	0	2	0
36	11.72	2	0	3	0	2	0	3	0	3	0
37	11.47	1	0	3	0	2	0	2	0	3	0
38	6.76	2	0	2	0	2	0	3	0	3	0
39	12.10	2	0	3	0	-	1	-	1	3	1
40	9.66	-	1	-	1	1	0	2	0	3	0
41	11.78	2	0	3	0	3	0	3	0	3	0
42	12.52	3	0	2	0	3	0	2	0	1	0
43	13.12	3	0	1	0	2	0	2	0	2	0
44	10.78	3	0	1	0	3	0	3	0	2	0
45	9.26	3	0	2	0	3	0	2	0	1	0
46	11.51	2	0	3	0	1	0	2	0	3	0
47	11.93	2	0	3	0	2	0	1	0	3	0
48	11.93	1	0	3	0	2	0	3	0	2	0
49	10.98	2	0	2	0	3	0	3	0	1	0
50	11.06	2	0	2	0	2	0	1	0	3	0
51	6.87	1	0	3	0	3	0	3	0	2	0
52	10.74	2	0	3	0	3	0	-	1	1	1
53	9.24	3	0	2	0	1	0	2	0	3	0
54	9.50	3	0	2	0	2	0	3	0	1	0
55	6.88	3	0	1	0	2	0	3	0	2	0
56	9.10	3	0	2	0	3	0	2	0	1	0
57	7.25	-	1	-	1	2	0	3	0	1	0
58	10.18	2	0	2	0	2	0	3	0	3	0
59	8.72	-	1	-	1	3	0	2	0	2	0
60	10.83	-	1	-	1	2	0	3	0	1	0
61	12.96	1	0	3	0	2	0	2	0	3	0
62	11.76	2	0	3	0	2	0	1	0	3	0
63	10.10	3	0	3	0	2	0	2	0	3	0
64	10.83	-	1	-	1	2	0	3	0	2	0
65	9.66	-	1	-	1	2	0	2	0	2	0
66	10.12	2	0	3	0	2	0	1	0	3	0
67	10.37	3	0	2	0	3	0	3	0	3	0
68	8.57	3	0	1	0	3	0	2	0	2	0
69	12.36	1	0	3	0	3	0	3	0	2	0
70	9.81	3	0	1	0	-	1	3	0	-	1
71	11.13	3	0	1	0	2	0	2	0	2	0
72	12.24	3	0	3	0	2	0	-	1	-	1
73	9.43	3	0	3	0	2	0	-	1	-	1
74	12.46	2	0	2	0	1	0	2	0	3	0
75	15.48	2	0	2	0	2	0	1	0	3	0
76	10.76	2	0	3	0	-	1	3	0	3	1
77	15.81	2	0	3	0	-	1	2	0	1	1
78	10.44	3	0	3	0	3	0	2	0	2	0
79	11.55	2	0	3	0	2	0	-	1	3	1
80	11.84	1	0	3	0	2	0	2	0	3	0
81	7.27	3	0	2	0	2	0	-	1	3	1
82	10.19	-	1	-	1	2	0	3	0	3	0
83	11.41	1	0	3	0	2	0	-	1	-	1
84	11.61	3	0	3	0	-	1	2	0	-	1
85	12.46	3	0	1	0	3	0	2	0	2	0
86	9.40	3	0	3	0	-	1	2	0	-	1
87	11.06	2	0	3	0	3	0	3	0	1	0
88	12.96	2	0	2	0	3	0	2	0	3	0
89	12.07	2	0	3	0	-	1	2	0	3	1
90	10.16	3	0	1	0	3	0	2	0	2	0
91	12.83	1	0	3	0	2	0	2	0	2	0
92	14.96	3	0	3	0	2	0	2	0	3	0
93	11.24	3	0	1	0	2	0	2	0	3	0
94	8.51	3	0	3	0	2	0	-	1	3	1
95	6.46	1	0	3	0	3	0	3	0	2	0
96	10.74	3	0	2	0	1	0	2	0	3	0
97	11.96	2	0	3	0	3	0	2	0	3	0
98	9.95	-	1	-	1	1	0	2	0	3	0
99	11.70	2	0	3	0	3	0	3	0	1	0
100	11.61	2	0	3	0	2	0	3	0	1	0

Table C.6: *Observed data of $n = 172$ computer hard-drive data (continues on the next page).*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3
1	0.018	1	0	2	0	2	0
2	0.036	1	0	2	0	2	0
3	0.043	-	1	2	0	-	1
4	0.045	1	0	2	0	2	0
5	0.074	-	1	-	1	-	1
6	0.119	1	0	2	0	2	0
7	0.131	-	1	-	1	-	1
8	0.143	-	1	2	0	-	1
9	0.171	1	0	2	0	2	0
10	0.261	-	1	2	0	-	1
11	0.316	1	0	2	0	2	0
12	0.334	1	0	2	0	2	0
13	0.368	2	0	1	0	2	0
14	0.475	1	0	2	0	2	0
15	0.484	2	0	1	0	2	0
16	0.489	2	0	1	0	2	0
17	0.494	-	1	2	0	-	1
18	0.594	1	0	2	0	2	0
19	0.604	1	0	2	0	2	0
20	0.664	2	0	1	0	2	0
21	0.697	1	0	2	0	2	0
22	0.712	1	0	2	0	2	0
23	0.743	1	0	2	0	2	0
24	0.743	2	0	1	0	2	0
25	0.749	-	1	-	1	-	1
26	0.752	2	0	1	0	2	0
27	0.789	2	0	1	0	2	0
28	0.831	1	0	2	0	2	0
29	0.833	2	0	2	0	1	0
30	0.867	-	1	-	1	-	1
31	0.874	2	0	1	0	2	0
32	0.890	-	1	2	0	-	1
33	0.955	2	0	2	0	1	0
34	1.030	-	1	2	0	-	1
35	1.040	2	0	2	0	1	0
36	1.100	2	0	2	0	1	0
37	1.110	1	0	2	0	2	0
38	1.170	-	1	-	1	-	1
39	1.210	-	1	-	1	-	1
40	1.300	1	0	2	0	2	0
41	1.310	1	0	2	0	2	0
42	1.310	2	0	2	0	1	0
43	1.330	2	0	2	0	1	0
44	1.350	-	1	-	1	-	1
45	1.350	-	1	-	1	-	1
46	1.370	1	0	2	0	2	0
47	1.420	2	0	2	0	1	0
48	1.460	2	0	2	0	1	0
49	1.490	1	0	2	0	2	0
50	1.530	1	0	2	0	2	0
51	1.570	-	1	2	0	-	1
52	1.580	-	1	2	0	-	1
53	1.580	-	1	2	0	-	1
54	1.610	-	1	2	0	-	1
55	1.610	-	1	-	1	-	1
56	1.630	-	1	-	1	-	1
57	1.640	2	0	2	0	1	0
58	1.660	2	0	2	0	1	0
59	1.670	-	1	-	1	-	1
60	1.680	1	0	2	0	2	0
61	1.680	2	0	2	0	1	0
62	1.680	2	0	1	0	2	0
63	1.710	2	0	2	0	1	0
64	1.720	2	0	2	0	1	0
65	1.730	1	0	2	0	2	0
66	1.750	2	0	1	0	2	0
67	1.820	2	0	2	0	1	0
68	1.880	-	1	2	0	-	1
69	2.000	-	1	2	0	-	1
70	2.030	-	1	-	1	-	1
71	2.040	-	1	2	0	-	1
72	2.040	2	0	2	0	1	0
73	2.050	1	0	2	0	2	0
74	2.050	1	0	2	0	2	0
75	2.070	-	1	2	0	-	1
76	2.090	2	0	2	0	1	0
77	2.120	2	0	2	0	1	0
78	2.150	2	0	1	0	2	0
79	2.180	2	0	2	0	1	0
80	2.220	2	0	2	0	1	0
81	2.230	1	0	2	0	2	0
82	2.260	2	0	2	0	1	0
83	2.260	-	1	2	0	-	1
84	2.270	-	1	-	1	-	1
85	2.290	-	1	-	1	-	1
86	2.300	1	0	2	0	2	0
87	2.350	-	1	-	1	-	1
88	2.370	-	1	-	1	-	1
89	2.390	1	0	2	0	2	0
90	2.400	2	0	1	0	2	0
91	2.410	-	1	2	0	-	1
92	2.410	2	0	2	0	1	0

Table C.7: *Continuation - observed data of $n = 172$ computer hard-drive data.*

System ID	t	δ_1	v_1	δ_2	v_2	δ_3	v_3
93	2.410	2	0	1	0	2	0
94	2.430	-	1	-	1	-	1
95	2.490	2	0	2	0	1	0
96	2.520	-	1	2	0	-	1
97	2.570	-	1	-	1	-	1
98	2.640	2	0	2	0	1	0
99	2.660	-	1	-	1	-	1
100	2.670	2	0	2	0	1	0
101	2.700	-	1	2	0	-	1
102	2.740	-	1	2	0	-	1
103	2.750	-	1	2	0	-	1
104	2.750	2	0	2	0	1	0
105	2.760	2	0	2	0	1	0
106	2.790	2	0	2	0	1	0
107	2.820	2	0	2	0	1	0
108	2.830	-	1	2	0	-	1
109	2.850	2	0	2	0	1	0
110	2.860	2	0	2	0	1	0
111	2.870	-	1	-	1	-	1
112	2.950	2	0	1	0	2	0
113	2.960	-	1	-	1	-	1
114	2.980	2	0	2	0	1	0
115	3.030	-	1	2	0	-	1
116	3.040	1	0	2	0	2	0
117	3.070	1	0	2	0	2	0
118	3.070	1	0	2	0	2	0
119	3.080	-	1	-	1	-	1
120	3.110	2	0	1	0	2	0
121	3.120	-	1	2	0	-	1
122	3.130	2	0	2	0	1	0
123	3.150	2	0	2	0	1	0
124	3.150	-	1	2	0	-	1
125	3.160	2	0	2	0	1	0
126	3.160	2	0	1	0	2	0
127	3.180	2	0	2	0	1	0
128	3.190	2	0	2	0	1	0
129	3.210	-	1	-	1	-	1
130	3.240	2	0	2	0	1	0
131	3.250	2	0	1	0	2	0
132	3.270	-	1	2	0	-	1
133	3.280	-	1	-	1	-	1
134	3.370	-	1	-	1	-	1
135	3.420	2	0	1	0	2	0
136	3.430	-	1	-	1	-	1
137	3.460	1	0	2	0	2	0
138	3.510	2	0	2	0	1	0
139	3.630	2	0	2	0	1	0
140	3.650	2	0	2	0	1	0
141	3.650	2	0	2	0	1	0
142	3.650	-	1	2	0	-	1
143	3.670	-	1	2	0	-	1
144	3.670	1	0	2	0	2	0
145	3.700	2	0	2	0	1	0
146	3.720	1	0	2	0	2	0
147	3.770	2	0	2	0	1	0
148	3.770	2	0	2	0	1	0
149	3.770	-	1	-	1	-	1
150	3.780	2	0	2	0	1	0
151	3.790	-	1	2	0	-	1
152	3.790	-	1	-	1	-	1
153	3.800	1	0	2	0	2	0
154	3.810	2	0	2	0	1	0
155	3.810	-	1	-	1	-	1
156	3.830	1	0	2	0	2	0
157	3.830	-	1	-	1	-	1
158	3.840	-	1	-	1	-	1
159	3.840	2	0	1	0	2	0
160	3.850	2	0	2	0	1	0
161	3.860	-	1	-	1	-	1
162	3.860	-	1	2	0	-	1
163	3.870	-	1	-	1	-	1
164	3.890	-	1	2	0	-	1
165	3.910	-	1	2	0	-	1
166	3.910	-	1	2	0	-	1
167	3.930	2	0	2	0	1	0
168	3.950	2	0	2	0	1	0
169	3.950	-	1	-	1	-	1
170	3.960	2	0	2	0	1	0
171	3.980	2	0	2	0	1	0
172	3.980	2	0	2	0	1	0

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