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Parallel systems using the Weibull model

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Abstract. A series system reliability is based on the minimum life time of its components. Its dual, the parallel system, is based on maximum. Here, we consider the statistical analysis of a parallel system where its components follows the Weibull parametric model. Our perspective is Bayesian. Due to the mathematical complexity, to obtain the posterior distribution we use the Metropolis-Hasting simulation method. Based on this posterior, we evaluated the evidence of the Full Bayesian Significance Test –FBST– for comparing component reliabilities. The reason for using FBST is the fact that we are testing precise hypotheses. An example illustrates the methodology.

Keywords: Metropolis-Hasting, evidence, significance test
PACS: 02.70.Rr

INTRODUCTION

We address the case of k (≥ 2) components parallel system. To handle this problem the idea is the dual of the one used for Bayesian analysis of series system. In other words, we deal with the 1 out of k reliability system. Recall that for the series system to work all k components must be working – k out of k. Our objective is the estimation of all parameters – of Weibull distributions – involved in the whole system.

Hence, consider a parallel system with k components and let X_j, $j = 1,\dots,k$, denote the failure time of the j-th component. The first assumption is that $X_1,\dots,X_k$ are statistically independent Weibull random variables. The parallel system fails only when all components fail. The observed random vector is then (T,δ): $T = \max(X_1,\dots,X_k)$ and $\delta = j$ if $T = X_j$, $j = 1,\dots,k$.

Consider now a sample of n such parallel system – independent and identically distributed systems. The set of n sample observations is $(\mathbf{T},\delta) = \{(T_i,\delta_i), i = i,\dots,n\}$. Note that, to obtain the sample we have considered for the i-th observation a vector $(X_{1i},\dots,X_{ki})$, $i = i,\dots,n$, of latent or invisible observations of all j components. In fact, we only record the maximum life time of all components of this i-th system. In addition, we record which component produce that maximum. For example, suppose that we have 1 out of 3 system and have the observation $(T = 59min, \delta = 2)$. That is, the second component was the last to fail and its failure time was 59 minutes. Although, the other two components fail before the second, their failure time could not be recorded.

For the purpose of this paper and simplicity we restrict our problem to two components parallel system, that is $k = 2$. For the general case of $k \geq 2$, we believe that a extention of our procedure would be straightforward.

The unknown distribution function of the j-th component is $F_j(t) = \Pr(X_j \leq t)$, for $j = 1,2$. Consequently the system distribution function is given by $F(t) = \Pr(T \leq t) = \Pr(\max(X_1,X_2) \leq t)$.

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The main objective of this paper is to derive a parametric Bayesian estimator of F_1 and $F2$ from the data $(\mathbf{T}, \delta)$ under the Weibull model.

In a series system problem with two components, Kaplan and Meier [1] derived the product-limit (PL) estimator for $S_j(\cdot) = 1 - F_j(\cdot)$, $j = 1,2$, the survival function of the first component, and showed that it is a maximum likelihood estimator. Breslow and Crowley [2] studied several properties of the PL estimator. Coque Jr. [3] developed the parametric estimator under Weibull model.

In a parallel system problem for $k = 2$, F_j plays the role of S_j in the series system. In a series system the observation is the minimum time of failure whereas in a parallel system it is the maximum. As long as we know, the first Bayesian statistical analysis for parallel system was Polpo [4]. In fact, it introduces a Bayesian nonparametric general solutions for statistical problems in parallel system. Now, it seems to be that here we for the first time introduce Weibull models in parallel systems.

In the next section we establish the likelihood function for the Weibull model and find the posterior distribution for its parameters. Then we show how to use the Metropolis-Hasting to perform the parameters estimation. In the sequel we present a simulated example to show how the estimation process works and we use the FBST to compare the component parameters based on the evidence given by Pereira and Stern [5]. At end we discuss our paper on the light of the statistical reliability literature.

We end this section with the notation used.

$f(t \mid \theta)$ density function with parameter vector θ at point t.
$F(t \mid \theta)$ distribution function with parameter vector θ at point t.
$L(\theta \mid t)$ likelihood function at point θ.
$\mathbb{I}(\cdot)$ unit function: $\mathbb{I}(TRUE) = 1$, $\mathbb{I}(FALSE) = 0$.
δ the last component to fail.
$F(\cdot)$ distribution function of the system.
$F_j(\cdot)$ distribution function of the j-th component.
$\max(a,b)$ maximum between a and b.
n sample size, number of systems observed.
$\Pr(E)$ probability of event E.
T system failure or survival time.
$(\mathbf{T}, \delta) = \{(T_i, \delta_i) : i = 1, \ldots, n\}$, random sample to be observed.
X_j j-th component failure time.

WEIBULL PARALLEL SYSTEM MODEL

Now we present the Weibull sample model to estimate each parameters of each component.

The likelihood function for the parallel system sample described in previous section is as follows:

$$L(\theta \mid \mathbf{T}, \delta) = \prod_{i:\delta_i=1} f_1(t_i \mid \theta) F_2(t_i \mid \theta) \prod_{i:\delta_i=2} f_2(t_i \mid \theta) F_1(t_i \mid \theta). \qquad (1)$$

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Considering now the Weibull model with standard parameters $\theta_j = (\beta_j, \eta_j)$, we obtain the following expressions:

$$f_j(t \mid \beta_j, \eta_j) = \frac{\beta_j}{\eta_j^{\beta_j}} t^{\beta_j - 1} \exp\left\{-\left(\frac{t}{\eta_j}\right)^{\beta_j}\right\},$$

$$F_j(t \mid \beta_j, \eta_j) = 1 - \exp\left\{-\left(\frac{t}{\eta_j}\right)^{\beta_j}\right\} \text{ and}$$

$$L(\beta_1, \eta_1, \beta_2, \eta_2 | \mathbf{T}, \delta) = \prod_{i:\delta_i=1} \left(\frac{\beta_1}{\eta_1^{\beta_1}}\right) t_i^{\beta_1 - 1} \exp\left\{-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right\}\left[1 - \exp\left\{-\left(\frac{t_i}{\eta_2}\right)^{\beta_2}\right\}\right]$$
$$\times \prod_{i:\delta_i=2} \left(\frac{\beta_2}{\eta_2^{\beta_2}}\right) t_i^{\beta_2 - 1} \exp\left\{-\left(\frac{t_i}{\eta_2}\right)^{\beta_2}\right\}\left[1 - \exp\left\{-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right\}\right].$$

Using Jeffrey's noninformative prior $\pi(\beta_1, \eta_1, \beta_2, \eta_2) \propto \frac{1}{\beta_1}\frac{1}{\eta_1}\frac{1}{\beta_2}\frac{1}{\eta_2}$ and the facts that the components are independent, the parameters are variation independent and the likelihood may be factored as a product of functions depending only on θ_1 or θ_2, the posterior as also factored on simple functions. Hence, the parameters θ_1 and θ_2 are independent both in the prior and in the posterior. The posterior density is as follows:

$$\pi(\beta_1, \eta_1 | \mathbf{T}, \delta) \propto \frac{1}{\beta_1}\frac{1}{\eta_1} \prod_{i:\delta_i=1} \left(\frac{\beta_1}{\eta_1^{\beta_1}}\right) t_i^{\beta_1 - 1} \exp\left\{-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right\} \prod_{i:\delta_i=2}\left[1 - \exp\left\{-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right\}\right].$$

Replacing 1 for 2 and 2 for 1 in the above expression, we obtain the posterior distribution for the parameters of the second component. Obviously there are no closed expression for both parameters, θ_1 and θ_2, posterior distribution functions. To obtain the estimation of the distribution functions we use the well known Metropolis-Hasting. This is describe in the next section.

ESTIMATION STEPS

For parameter estimation we use the posterior mean, although there is no closed form for it. The Metropolis-Hasting method should be appropriate for our problems.

The starting distribution for this simulation method used here was the Gamma distribution for the parameters θ_1 and $\theta 2$. The obtained posterior distributions from this method was used to obtain the estimates (the means of these distributions) of the parameters.

Next section presents an example to illustrate the use of the procedure described above. The convergence of the method is studied with ergodic means and autocorrelation graphics.

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TABLE 1. Point estimation.

	β_1	η_1	β_2	η_2
true value	4	2	2	3
estimated (sd)	5.177 (0.808)	2.119 (0.072)	1.931 (0.215)	2.832 (0.191)

EXAMPLE: SIMULATED OBSERVATIONS

The following example is based in 100 simulated observations from de following parallel system: X_1 and X_2 are distributed as Weibull with parameters $\theta_1 = (\beta_1, \eta_1) = (4,2)$ and $\theta_2 = (\beta_2, \eta_2) = (2,3)$.

The goal of such example is to evaluate the quality of the Bayesian estimation. Since we have fixed the distributions, we can check how good are the estimates comparing them with the true values of the parameters. From the sample of size 100 we observed 64 component one as maximum and 36 of component two.

Figure 1 shows that the process converges fast, so we use a burn-in of size 5000. Figure 2 shows that for all parameters the auto-correlation is low, hence we could use a jump of only 5 simulated points of the posterior. The number of generate points from the Metropolis-Hasting method to built the posterior distributions was 10000 for each parameter.

Figure 3 and Figure 4 present the distribution function estimates for component 1 and 2. Figure 5 illustrates how good is the system estimate even with the presence of censored data. We have censored the failure time of the component that had the minimum observation.

Table 1 list the estimates and the standard deviation of each parameter. The numbers reforce the quality of our estimation method.

To evaluate the evidence index from FBST procedure we compute probability of the tangential set to the hypothesis $\mathbf{H}_0$: $\begin{cases} \beta_1 = \beta_2 \\ \eta_1 = \eta_2 \end{cases}$.

A set $\mathscr{T}$ is said to be tangential to $\mathbf{H}_0$ if all his points have density higher than the density of any parameter point satisfying $\mathbf{H}_0$. The evidence in favor is $1 - \Pr(\mathscr{T})$. In our case we obtain $EV[\mathbf{H}_0] \simeq 0$. In this case the evidence favors alternative hypothesis, rejecting $\mathbf{H}_0$ as should be.

FINAL REMARKS

In this paper we present a solution for a parallel system statistical analysis. We consider a very simple example to well illustrated the force of the method. Clearly, we used the corrected Weibull distribution, the same that generates the observations. However, most of the time the underline distribution is not known. The Weibull distribution is a very rich one since it may represent many types of distribution forms. There is a version of the Weibull distribution with three parameters which makes it still better to represent most situations.

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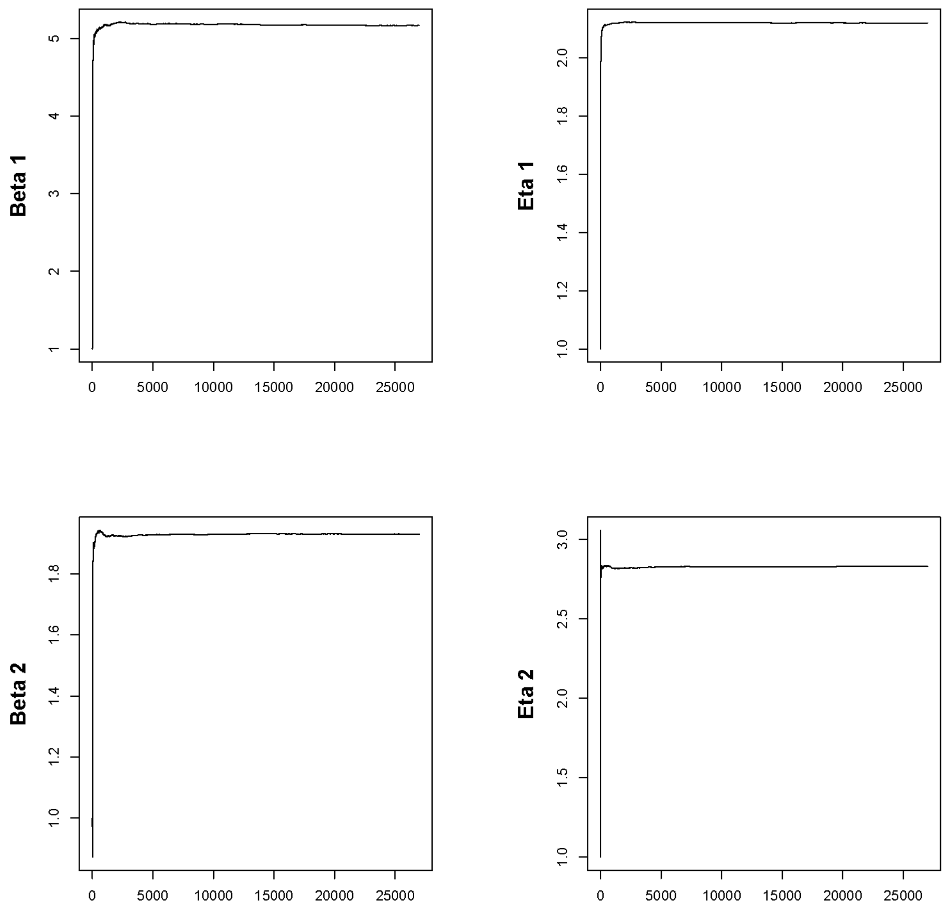

FIGURE 1. Ergodic Mean.

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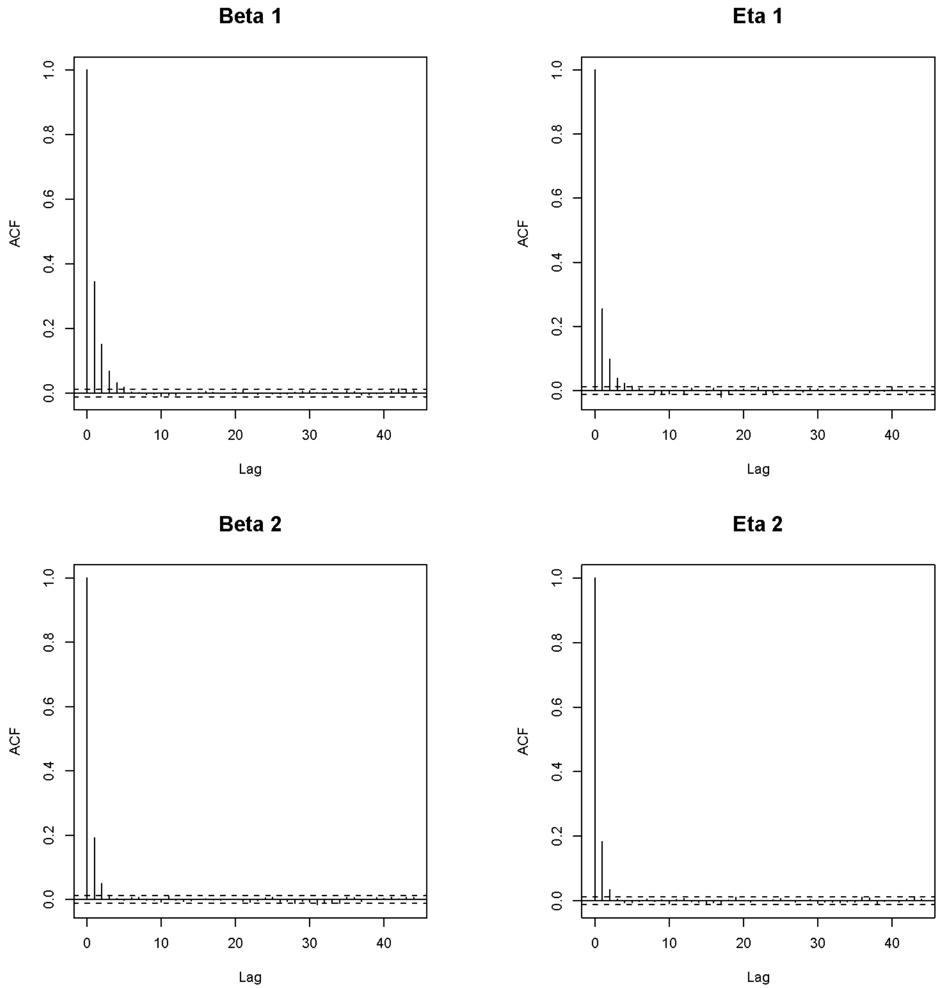

FIGURE 2. Auto-correlations.

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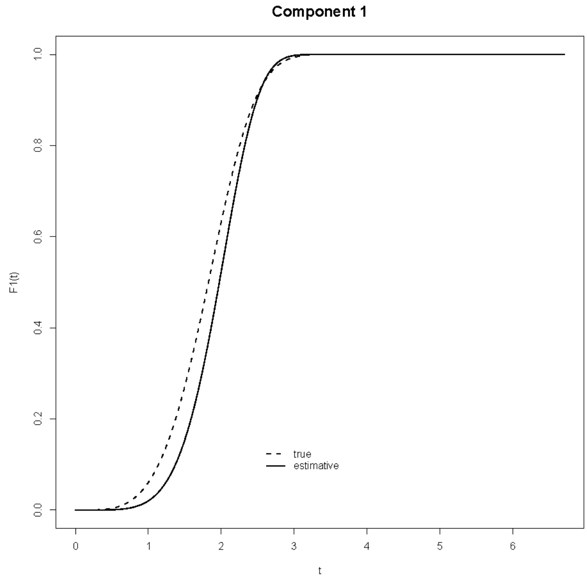

FIGURE 3. Component 1 distribution function estimate.

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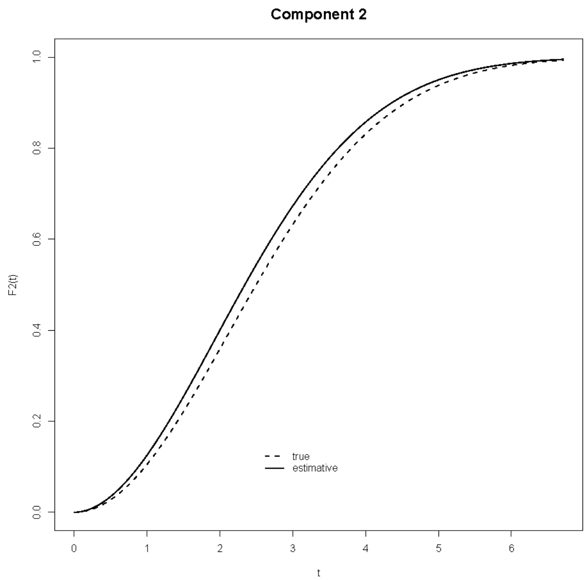

FIGURE 4. Component 2 distribution function estimate.

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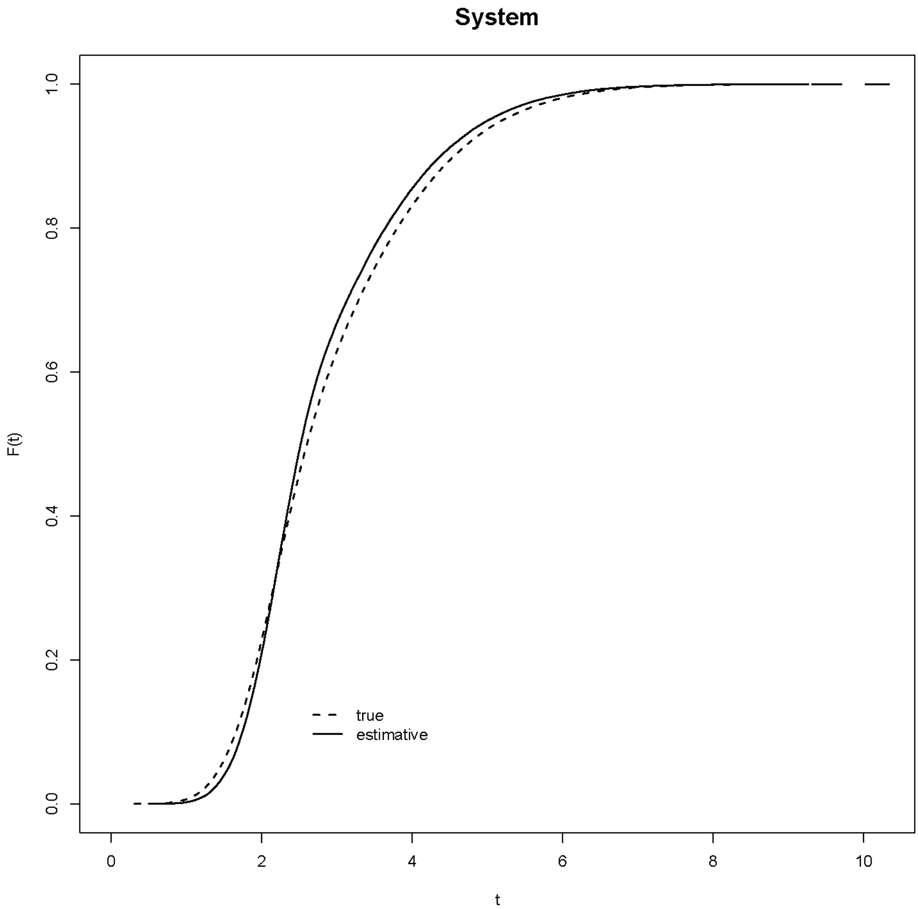

FIGURE 5. System distribution function estimate.

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