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BLACKWELL SUFFICIENCY AND BERNOULLI EXPERIMENTS

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Summary

The intuition behind Blackwell sufficiency concept is discussed using influence diagrams. A simple geometrical solution for the problem of comparing Bernoulli experiments is presented.

Key words: Informative experiments; sufficient experiment; transition function for sample spaces.

1. Introduction

Let x and y be two Bernoulli experiments (random variables) with unknown parameters π and $\pi/2$ $(0 < \pi < 1)$ respectively. In order to gain information about π, the following question is relevant: "Which one, x or y, is more informative for π ?" Usually we obtain the answer "It depends on the value of π". People who give this answer have in mind the inverse of the variance, that is, Fisher's information. However, we will show that x must be at least as informative as y, independent of the value of π.

By first performing the Bernoulli experiment x and then tossing a fair coin and letting $r = 1$ ($r = 0$) if a tail (head) occurs, we can compute $y^* = xr$. In this case y^* is a copy of y in the sense that y and y^* have the same distribution. By performing this simple randomization exercise we have obtained a copy of y using x. This means that x is Blackwell sufficient for y. [This is the definition in Section 2 We show in Section 3 that y is not Blackwell sufficient for x. That is, we cannot obtain a copy of x by using y. In fact, we will conclude that x is more informative for π than y. Note that using Fisher information, this result does not follow. Information here is about the unknown parameter, π, and is contained in the experiment.

2. Blackwell sufficiency

A statistical experiment related to a parameter $\pi \in \Pi$ (Π is a general parameter space) is an observable random quantity, x, associated with a sample space X and a family of probability functions (distributions) on X indexed by π, $(p_\pi;\ \pi \in \Pi)$. We avoid all measurability difficulties by restricting ourselves to discrete sample spaces. Given two sample spaces, X and Y, a transition function, F, from X to Y, is a family $F = \{f_x;\ x \in X\}$ of probability functions, $f_x(\cdot)$, defined on Y and indexed by x. For example, the family of hypergeometric probability functions,

$$f_x(Y) = \frac{\binom{x}{y}\binom{N-x}{n-y}}{\binom{N}{n}},$$

is a transition function from $(0, 1, \ldots, N)$ to $(0, 1, \ldots, n)$.

Let x and y be two experiments with models $[X, \{p_\pi;\ \pi \in \Pi\}]$ and $[Y, \{q_\pi;\ \pi \in \Pi\}]$, respectively.

Definition: (Blackwell) Experiment x is *sufficient for (at least as informative as)* experiment y, write $x \gg y$, if there exists a transition function

$F = \{f_x(\cdot);\, x \in X\}$ from X to Y such that

$$q_\pi(y) = \sum_{x \in X} f_x(y) p_\pi(x) \tag{2.1}$$

for all $y \in Y$ and $\pi \in \Pi$. (We may also write $y \ll x$ to say that y is at most as informative as x.)

A transition function F satisfying (2.1) is called a Blackwell transition function (it does not depend on the value of π). It is no difficult to check that he relation $\gg$ defines a partial ordering on the family of experiments related to π.

If $y = y(x)$ is a sufficient statistic in the classical sense of Fisher (i.e., the conditional distribution of x given y does not involve π), then it follows at once that y is sufficient for x in Blackwell's sense ($y \gg x$). Of course, x is sufficient for y in either sense. If $x \gg y$ and $x \ll y$ we write $x \approx y$ to indicate that x and y are equally informative.

The intuitive content of the relation "$\gg$" is as follows: Let x be Blackwell sufficient for y, $x \gg y$, and let the transition function be $f_x(y)$. If we perform experiment x, record its outcome and carry out a post-randomization exercise that chooses a point in the sample space Y in accordance with the probability function f_x, then the result of the experiment, y^*, is in a sense indistinguishable from y in that both are endowed with the same model $[Y, \{q_\pi;\, \pi \in \Pi\}]$. Any decision rule related to π which is based on the experiment resulting in y can therefore be perfectly matched (in terms of their average performance characteristics) by a randomized rule using x.

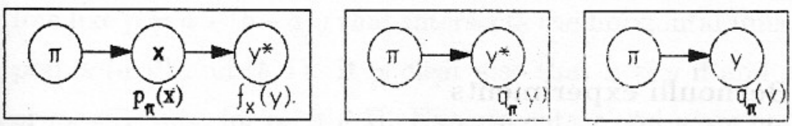

Figure 1

Blackwell sufficiency illustrated by influence diagrams.

Figure 1 (Shachter, 1988, personal communication with the second author) shows three influence diagrams relating π, x, and y^* and π and y. The first shows the dependence of x on the value of π and the (conditional) independence of y^* on the value of π when the value of x is known. Note that there is no arc connecting π to y^*. The second diagram shows the dependence of y^* on the value of π after the elimination of x from the graph; i.e., it shows the marginal distribution of y^*. The third diagram shows the dependence of y on the value of π. The second and the third graph being equivalent means that $x \gg y$. To say that there is a Blackwell transition function from X to Y is to say that there is no arc from π to y^* in the first diagram of Figure 1. (For a first introduction to influence diagrams see Barlow, 1989.)

For the example of Section 1, we write $x \sim \text{Ber}(\pi)$ and $y \sim \text{Ber}\left(\frac{\pi}{2}\right)$ for the two experiments and $r \sim \text{Ber}\left(\frac{1}{2}\right)$ for the randomized rule. Writing $y^* = xr$ then, y^* has the same distribution of y proving that $x \gg y$. The Blackwell transition function in this case is defined by $f_1(1) = \Pr(y^* = 1|x = 1) = \frac{1}{2}$ and $f_0(0) = \Pr(y^* = 0|x = 0) = 1$. Figure 2 illustrates this solution.

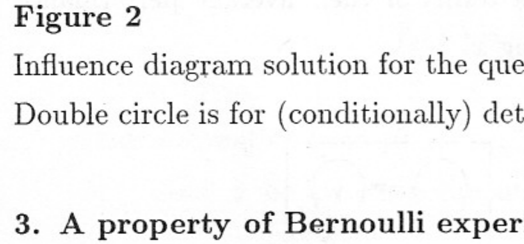

Figure 2

Influence diagram solution for the question of Section 1.

Double circle is for (conditionally) deterministic nodes.

3. A property of Bernoulli experiments

In this section, a simple geometrical result is presented. It permits one to check whether two Bernoulli experiments are comparable in Blackwell's sense.

Were they comparable, a simple rule to identify the most informative experiment is derived. Let $x \sim \text{Ber}(p_\pi)$ and $y \sim \text{Ber}(q_\pi)$ be two Bernoulli experiments related to the same (arbitrary) parameter $\pi \in \Pi$.

Proposition. *x and y are comparable in the sense of Blackwell sufficiency if and only if the set $\{(p_\pi, q_\pi);\ \pi \in \Pi\}$ lies on a linear segment that intersects two opposite sides of the unit square: $x \gg y$ ($x \ll y$) if the line intersects the vertical (horizontal) sides. Consequently, if the line is diagonal then, $x \approx y$.*

Proof: A transition function from $\{0,1\}$ to $\{0,1\}$ is a stochastic matrix

$$f = \begin{pmatrix} 1-a & a \\ 1-b & b \end{pmatrix}, \qquad 0 < a < 1, \quad 0 < b < 1. \tag{3.1}$$

Such an f can transform x into an experiment like y if and only if

$$q_\pi = (1-p_\pi)a + p_\pi b = a + (b-a)p_\pi,$$

for all $\pi \in \Pi$. Thus, $x \gg y$ if and only if the set $\{(p_\pi, q_\pi);\ \pi \in \Pi\}$ lies on a line $q = a + (b-a)p$ that joins the points $(0,a)$ and $(1,b)$ on the vertical sides of the unit square. Similarly, $x \ll y$ if and only if the set $\{(p_\pi, q_\pi);\ \pi \in \Pi\}$ lies on a line like $p = a + (b-a)q$ that intersects the horizontal lines of the unit square at points $(a,0)$ and $(b,1)$. It is clear also that $x \approx y$ if and only if either $p_\pi = q_\pi$ or $p_\pi = 1 - q_\pi$, for all $\pi \in \Pi$. Experiments x and y are not comparable if either the points in the set $\{(p_\pi, q_\pi);\ \pi \in \Pi\}$ are not collinear or if they lie on a line that intersects two adjacent sides of the unit square.

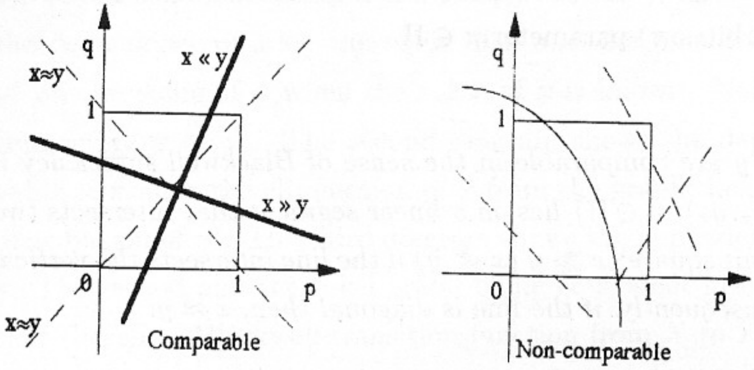

Figure 3

Comparing Bernoulli experiments in the Blackwell sense.

Returning to the example of Section 1, it is easily seen that the set

$$\{(p_\pi, q_\pi);\ \pi \in \Pi\} = \left\{(\pi, \frac{\pi}{2});\ 0 < \pi < 1\right\}$$

lies on the line $q = \frac{1}{2}p$. It is interesting to note that $p_\pi = \pi$ takes values in the whole unit interval and that $q_\pi = \frac{\pi}{2}$ is restricted to the interval $\left(0, \frac{1}{2}\right)$. A simple extension is the case where for a positive real number c we have

$$x \sim \text{Ber}(\pi) \quad \text{and} \quad y \sim \text{Ber}(c\pi).$$

In this case, if $c < 0$ $(c > 0)$ then, $x \gg y$ $(x \ll y)$. Also, $x \approx y$ if and only if $c = 1$.

4. A classical example

The fruitfulness of the notion of Blackwell sufficiency is best exemplified by the following simple situation: Consider a population of individuals that are categorized in a two-way tabular form in terms of two attributes E and F (we write E' and F' for their complements) as follows:

Table 1

Population frequencies where

π is the unknown quantity of interest.

	E	E′	sum
F F′	π e-π	f-π 1-e-f+π	f 1-f
sum	e	1-e	1

The only unknown parameter of interest is $\Pr(EF) = \pi$. The marginal relative frequencies (marginal probabilities) $\text{pr}(E) = e$ and $\Pr(F) = f$ are known. To seek information about π we can draw an individual randomly from the whole population and observe whether this sample unit belongs to EF or not, thus simulating a Bernoulli experiment, $x \sim \text{Ber}(\pi)$. Alternatively, we may sample from attribute E (assume that a sampling frame for the sub-population E is available) and observe if EF occurs or not, thus simulating an experiment $x_E \sim \text{Ber}\left(\frac{\pi}{e}\right)$. Similarly, the following alternative experiments can be performed:

$$x_F \sim \text{Ber}\left(\frac{\pi}{f}\right), \; x_{E'} \sim \text{Ber}\left(\frac{f-\pi}{1-e}\right) \quad \text{and} \quad x_{F'} \sim \text{Ber}\left(\frac{e-\pi}{1-f}\right).$$

Which of the five experiments (if any), x, x_E, x_F, $x_{E'}$ and $x_{F'}$, is the most informative for π? Assume without loss of generality that

$$0 < e < f < 1-f < 1-e < 1.$$

The answer to this question is (Blackwell & Girshick, 1954) "x_E is more informative than all the other four experiments". The proposition of Section 3 illuminates this interesting situation.

Note that we have the following sets with their corresponding lines intersecting the vertical opposite sides of the unit square:

(1) $\left\{\left(\frac{\pi}{e}, \pi\right); 0 < \pi < e\right\}$ lies on $q = ep$ showing that $x_E \gg x$,

(2) $\left\{(\frac{\pi}{e}, \frac{\pi}{f}); 0 < \pi < e\right\}$ lies on $q = \frac{e}{f}p$ showing that $x_E \gg x_F$,

(3) $\left\{(\frac{\pi}{e}, \frac{f-\pi}{1-e}); 0 < \pi < e\right\}$ lies on $q = \frac{f}{1-e} - \frac{e}{1-e}p$ showing that $x_E \gg x_{E'}$,

(4) $\left\{(\frac{\pi}{e}, \frac{e-\pi}{1-f}); 0 < \pi < e\right\}$ lies on $q = \frac{e}{1-f} - \frac{e}{1-f}p$ showing that $x_E \gg x_{F'}$,

(5) $\left\{(\frac{\pi}{f}, \pi); 0 < \pi < f\right\}$ lies on $q = fp$ showing that $x_F \gg x$,

(6) $\left\{(\frac{\pi}{f}, \frac{f-\pi}{1-e}); 0 < \pi < f\right\}$ lies on $q = \frac{f}{1-e} - \frac{f}{1-e}$ showing that $x_F \gg x_{E'}$, and

(7) $\left\{(\frac{e-\pi}{1-f}, \frac{f-\pi}{1-e}); 0 < \pi < 1-f\right\}$ lies on $q = \frac{f-e}{1-e} - \frac{1-f}{1-e}p$ showing that $x_{F'} \gg x_{E'}$.

This proves that, among the five experiments, x_E is the most informative. In the same manner it is proved that $x_F \gg x$, $x_F \gg x_{E'}$, and $x_{F'} \gg x_{E'}$. It is interesting, however, to see that for each of the remaining 3 pairs, $[x_F, x_{F'}]$, $[x, x_{E'}]$ and $[x, x_{F'}]$, their elements are not comparable in Blackwell's sense. The following sets with their corresponding lines, that intersect adjacent sides of the unit square, show this phenomenon:

(8) $\left\{(\frac{\pi}{f}, \frac{e-\pi}{1-f}); 0 < \pi < f\right\}$ lies on $q = \frac{e}{1-f} - \frac{f}{1-f}p$ showing that x_F and $x_{F'}$ are not comparable,

(9) $\left\{(\pi, \frac{f-\pi}{1-e}); 0 < \pi < 1\right\}$ lies on $q = \frac{1}{1-e} - \frac{1}{1-e}p$ showing that x and $x_{E'}$ are not comparable, and

(10) $\left\{(\pi, \frac{e-\pi}{1-f}); 0 < \pi < 1\right\}$ lies on $q = \frac{e}{1-f} - \frac{1}{1-f}p$ showing that x and $x_{F'}$ are not comparable.

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