Nonparametric Bayesian Estimation of Reliabilities in a Class of Coherent Systems

Adriano Polpo, *Member, IEEE*, Debajyoti Sinha, and Carlos A. de B. Pereira

***Abstract*—Usually, methods evaluating system reliability require engineers to quantify the reliability of each of the system components. For series and parallel systems, there are limited options to handle the estimation of each component's reliability. This study examines the reliability estimation of complex problems of two classes of coherent systems: series-parallel, and parallel-series. In both of the cases, the component reliabilities may be unknown. We developed estimators for reliability functions at all levels of the system (component and system reliabilities). The main assumption required is that, for all the distributions of the components of a particular system, the sets of discontinuity points have to be disjoint. Nonparametric Bayesian estimators of all sub-distribution and distribution functions are derived, and a Dirichlet multivariate process as a prior distribution is considered for the nonparametric Bayesian estimation of all distributions. For illustration, two simulated numerical examples are presented. The estimators are s-consistent, and one may observe from the examples that they have good performance. Our estimator can accommodate continuous failure distributions, as well as distributions with mass points.**

***Index Terms*—Coherent systems, Dirichlet multivariate processes, reliability theory, series-parallel systems.**

Acronyms

CHR	cumulative hazard rate
CRHR	cumulative reversed hazard rate
DF	distribution function
HR	hazard rate
MAE	mean absolute error
RHR	reversed hazard rate
PSS	parallel-series system
SD	standard deviation
SDF	sub-distribution function
SPS	series-parallel system

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A. Polpo is with the Department of Statistics, Federal University of São Carlos, São Carlos, Brazil (e-mail: polpo@ufscar.br).

D. Sinha is with Department of Statistics, Florida State University, Florida, USA (e-mail: sinhad@stat.fsu.edu).

C. A. d. B. Pereira is with the Department of Statistics, University of São Paulo, São Paulo, Brazil (e-mail: cpereira@ime.usp.br).

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Notation

$\mathrm{Beta}(a,b)$	distribution Beta with parameters a and b.
$\mathcal{D}_k(\alpha_1,\ldots,\alpha_k)$	Dirichlet multivariate (k-variate) process with parameters $\alpha_1,\ldots,\alpha_k$.
$DM_k(\alpha_1,\ldots,\alpha_k)$	Dirichlet multivariate distribution with parameters $\alpha_1,\ldots,\alpha_k$.
δ	the last component to fail (that is, if $\delta=j$, then the system has failed because of the j-th component, X_j).
d_{ji}	$=\sum_{\ell=1}^{n}\mathbb{I}(T_\ell=T^{\bullet}_{(i)},\delta_\ell=j)$.
$F(\cdot)$	distribution function of the system.
$F_j(t)$	$=\Pr(X_j\le t)$, the distribution function of the j-th component.
$F_j^*(t)$	$=\Pr(T\le t,\delta=j)$, the sub-distribution function of the j-th component.
$F_{jn}^*(t)$	$=(1/n)\sum_{i=1}^{n}\mathbb{I}(T_i\le t,\delta=j), j=1,\ldots,k$, the empirical sub-distribution function of the j-th component.
$F_n(t)$	$=\sum_{j=1}^{k}F_{jn}^*(t)$, the empirical distribution function of the system
$\mathbb{I}(\cdot)$	unit function: $\mathbb{I}(\mathrm{TRUE})=1$, $\mathbb{I}(\mathrm{FALSE})=0$.
$\prod$	product-integral.
$\oint g(s)\mathrm{d}s$	integration over disjoint open intervals that do not include the jump points of $g(\cdot)$.
jump point	of $F(\cdot)$ is any point t such that $F(t^-)<F(t)$, where $F(\cdot)$ is a distribution function.
$\lambda(\cdot)$	hazard rate.
$\Lambda(\cdot)$	cumulative hazard rate.
$\mu(\cdot)$	reversed hazard rate.
$\mathcal{M}(\cdot)$	cumulative reversed hazard rate.
MAE	$=\mathrm{MAE}(F_a,F_b)=(1/n)\sum_{i=1}^{n}\lvert F_a(t_i)-F_b(t_i)\rvert$.
$\max(a,b)$	maximum between a and b.
$\min(a,b)$	minimum between a and b.

n	number of systems in the sample.
N_i	$= \sum_{\ell=1}^{n} \mathbb{I}(T_\ell < T^\bullet_{(i)})$.
$\Pr(E)$	probability of event E.
$\prod^G$	product over jump points of $G(\cdot)$.
ρ_j	reliability of the j-th component.
set of jump points	of $F(\cdot)$ is an enumerate (discrete) set of all jump points of the distribution function F.
support	of a random variable T is the sample space Ω_T of T, that is, $\Omega_T = \{t : 0 < F(t) < 1\}$, where $F(\cdot)$ is the distribution function of T.
T	the system failure or survival time.
$(\mathbf{T}, \boldsymbol{\delta})$	$= \{(T_i, \delta_i) : i = 1, \ldots, n\}$, random sample to be observed.
$T^\bullet_{(i)}$	i-th distinct order statistics.
X_j	j-th component failure time.

I. Introduction

SYSTEM reliability of the coherent structure has been studied by many authors [1]–[6], who either consider the component's reliability to be known, or the system to be multi-stage [7]–[9]. In this study, we have developed a nonparametric estimator for all the reliability functions involved in the series-parallel system and parallel-series system under the assumptions that the components' reliabilities are unknown; the only available information are the failure times of the system and the component that produced the failure, and the notion that two components cannot fail at same instant of time. It must be noted that this is a very challenging problem because the failure times of the components involved in a system are mostly censored by the failure of other component. Moreover, here, we are not limited to only the right-censor or left-censor, because in a series-parallel system (SPS) or parallel-series system (PSS), we can either observe the exact failure time of the component, or the failure time can be right-censored or left-censored, which makes the components' reliability estimation problem of the SPS (or the PSS) a very difficult task. It is very common to have 80%–90% of the failure time data of the components be censored. However, the system does not have censored data, so it is not complicated to estimate the system's reliability, and hence, our main interest is in the estimation of the components' reliabilities.

The reliability estimation problem when the distributions of the components are unknown has received many contributions in recent decades. Some important references in this subject are as follows. Langberg *et al.* [10] presented one form to convert s-dependent models into s-independent ones, using the assumption that two components cannot fail at the same time. Peterson [11], and Tsiats [12] developed some important results with respect to nonparametric estimation of the series system (competing risks), while Salinas-Torres *et al.* [13] developed a Bayesian nonparametric estimator, which was corrected by Polpo and Sinha [14]. Furthermore, Polpo and Pereira [15] reported similar results for the problem of parallel systems (co-operating system) to those presented in [11], and [13]. Also, for Bayesian nonparametric analysis, in [16] the Dirichlet process was reported, and in [17] the multivariate Dirichlet process has been described.

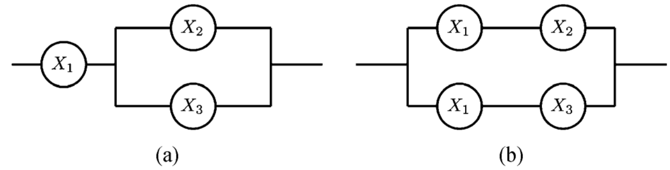


Fig. 1. (a) SPS; (b) PSS representation for the SPS in Fig. 1(a).

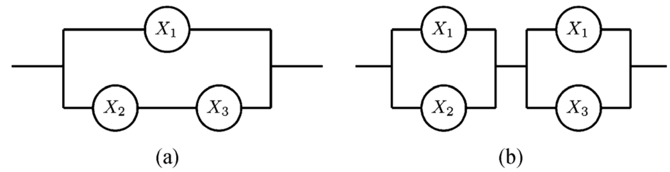


Fig. 2. (a) PSS; (b) SPS representation for the PSS in Fig. 2(a).

It is known that any coherent system can be written as a SPS or a PSS (Barlow and Proschan [1]). Using their results, the SPS (Fig. 1(a)) can be represented as a PSS (Fig. 1(b)), and the PSS (Fig. 2(a)) can be represented as a SPS (Fig. 2(b)). However, it is very common that, in the representation of the system, some components appear in two different places within it. For example, consider the component X_1 in Fig. 1(b) (or Fig. 2(b)). We have the reliabilities of four components to estimate. However, two of them are in fact the same component X_1, and will fail at the same time, which violates our assumptions. For this reason, it is important to have the estimators for both the SPS and the PSS that give a wide variety of representations. If one of these representations does not violate our assumptions, then our proposed estimator can be used there.

In Section II, we have given the probability results necessary for the development of the estimator. Section III is devoted to the construction of the nonparametric Bayesian estimator for the SPS and the PSS with three components $(k = 3)$. In Section IV, we have extended the results to a more general case of $k \geq 4$; and in Section V, we have shown how to use the proposed estimator, and illustrated its qualities. Last, in Section VI, we have presented some final comments and possible future works. The proofs of the theorems are given in the Appendix.

II. Probability Relations

In this section, we present the important results and properties of the SPS and the PSS. We restrict ourselves to a system with three components $(k = 3)$, given in Fig. 1(a), and in Fig. 2(a). It must be noted that, with two components $(k = 2)$, it is only possible to have a series or parallel system.

Let (X_1, X_2, X_3) be the lifetimes of three components of a SPS (Fig. 1(a)) or a PSS (Fig. 2(a)), with marginal distribution functions (DF) F_1, F_2, and F_3, respectively. The indicator of the component that produced the system failure is $\delta = 1$ when $T = X_1$, $\delta = 2$ when $T = X_2$, and $\delta = 3$ when $T = X_3$.

The restriction here is that the three sets of jump points of F_1, F_2, and F_3 must be disjoint. The following properties can be proved.

Property 1: The sub-distribution functions (SDF) F_1^*, F_2^*, and F_3^* determine the DF of the system,

$$F(t) = F_1^*(t) + F_2^*(t) + F_3^*(t). \tag{1}$$

Property 2:

1) $F_1^*(+\infty) = \Pr(\delta = 1)$;
2) $F_2^*(+\infty) = \Pr(\delta = 2)$;
3) $F_3^*(+\infty) = \Pr(\delta = 3)$;
4) $F_1^*(+\infty) + F_2^*(+\infty) + F_3^*(+\infty) = 1$.

Property 3: The set of jump points F_j^* and F_j are the same, where $j = 1,2,3$. Because F_1, F_2, and F_3 have disjoint set of jump points, so have F_1^*, F_2^*, and F_3^*.

Property 4: If $\min(F_1(t), F_2(t), F_3(t)) < 1$ for $t < t^*$, and 1 for $t \geq t^*$, then t^* is the largest support point of the system.

The lifetime of the system is $T = \min(X_1, \max(X_2, X_3))$, and the system reliability of s-independent components is

$$R(t) = [1 - F_1(t)]\,[1 - F_2(t)F_3(t)]\,, \tag{2}$$

for the SPS, and

$$R(t) = 1 - F_1(t)\,\{1 - [1 - F_2(t)]\,[1 - F_3(t)]\}\,, \tag{3}$$

for the PSS.

Property 5: The SDF of the SPS can be expressed using the marginal DF of the components by

$$\begin{aligned} F_1^*(t) &= \int_0^t [1 - F_2(t)F_3(t)]\,\mathrm{d}F_1(t), \\ F_2^*(t) &= \int_0^t [1 - F_1(t)]\,F_3(t)\mathrm{d}F_2(t), \\ F_3^*(t) &= \int_0^t [1 - F_1(t)]\,F_2(t)\mathrm{d}F_3(t), \end{aligned} \tag{4}$$

and the SDF of the PSS can be expressed using the marginal DF of the components by

$$\begin{aligned} F_1^*(t) &= \int_0^t 1 - [1 - F_2(t)]\,[1 - F_3(t)]\,\mathrm{d}F_1(t), \\ F_2^*(t) &= \int_0^t F_1(t)\,[1 - F_3(t)]\,\mathrm{d}F_2(t), \\ F_3^*(t) &= \int_0^t F_1(t)\,[1 - F_2(t)]\,\mathrm{d}F_3(t). \end{aligned} \tag{5}$$

Our interest is to obtain the inverse of (4) and (5); that is, to express the DF F_2 as a function of the SDF (F_1^*, F_2^*, F_3^*). This inverse is presented with the following definitions and theorems.

Definition 1: The functions $\Phi_s(F_1^*, F_2^*, F_3^*, t)$, and $\Phi_p(F_1^*, F_2^*, F_3^*, t)$ based on sub-distributions F_1^*, F_2^*, and F_3^* are

$$\Phi_s(F_1^*, F_2^*, F_3^*, t) \equiv 1 - \left\{ \exp\left[\oint_0^t \frac{-\mathrm{d}F_1^*(v)}{1 - \sum_{j=1}^{3} F_j^*(v)} \right] \prod_{v \leq t}^{F_1^*} \left[\frac{1 - \sum_{j=1}^{3} F_j^*(v^+)}{1 - \sum_{j=1}^{3} F_j^*(v^-)}. \right] \right\},$$

$$\Phi_p(F_1^*, F_2^*, F_3^*, t) \equiv \exp\left\{ \oint_t^\infty \frac{-\mathrm{d}F_1^*(v)}{\sum_{j=1}^{3} F_j^*(v)} \right\} \prod_{v > t}^{F_1^*} \left[\frac{\sum_{j=1}^{3} F_j^*(v^-)}{\sum_{j=1}^{3} F_j^*(v^+)} \right].$$

The functions Φ_s (for a series system), and Φ_p (for a parallel system) are the versions with three components for those presented in [14], and [15], respectively. First, Theorem 1 states the relation between F_1 and F_1^*, F_2^*, and F_3^*. The functions Φ_s, and Φ_p can be used to define the relations between the DF F_2 and the sub-distributions F_1^*, F_2^*, and F_3^* in a series system, or in a parallel system, respectively.

Theorem 1: The SDF F_1^*, F_2^*, and F_3^* determine (uniquely) the DF F_1 of a SPS for $t \leq t^*$ by $F_1(t) = \Phi_s(F_1^*, F_2^*, F_3^*, t)$, and the DF F_1 of a PSS for $t \leq t^*$ by $F_1(t) = \Phi_p(F_1^*, F_2^*, F_3^*, t)$.

The next definition gives the functions Φ_{sp} (for the SPS), and Φ_{ps} (for the PSS) that are the inverses of (4), and (5), respectively. These two functions are the ones that relate from what we can observe from a sample (F_1^*, F_2^*, and F_3^*) to what we want to know (F_2). Based on these functions, and on the Dirichlet process, we have developed the Bayesian nonparametric estimator for the components involved in the SPS (or the PSS).

Definition 2: The functions $\Phi_{sp}(F_1^*, F_2^*, F_3^*, t)$, and $\Phi_{ps}(F_1^*, F_2^*, F_3^*, t)$, based on sub-distributions F_1^*, F_2^*, and F_3^*, are

$$\begin{aligned} \Phi_{sp}\,(F_1^*, F_2^*, F_3^*, t) &\equiv \exp\left\{ \oint_t^\infty \frac{-\mathrm{d}F_2^*(v)}{\sum_{j=1}^{3} F_j^*(v) - \Phi_s\,(F_1^*, F_2^*, F_3^*, v)} \right\} \\ &\quad \times \prod_{v>t}^{F_2^*} \left[\frac{\sum_{j=1}^{3} F_j^*(v^-) - \Phi_s\,(F_1^*, F_2^*, F_3^*, v^-)}{\sum_{j=1}^{3} F_j^*(v^+) - \Phi_s\,(F_1^*, F_2^*, F_3^*, v^+)} \right], \end{aligned}$$

$$\Phi_{ps}\,(F_1^*, F_2^*, F_3^*, t) \equiv 1 - \left\{ \exp\left[\oint_0^t \frac{-\mathrm{d}F_2^*(v)}{\Phi_p\,(F_1^*, F_2^*, F_3^*, v) - \sum_{j=1}^{3} F_j^*(v)} \right] \right.$$

$$\times \prod_{v \leq t}^{F_2^*} \left[\frac{\Phi_p\left(F_1^*, F_2^*, F_3^*, v^+\right) - \sum_{j=1}^{3} F_j^*(v^+)}{\Phi_p\left(F_1^*, F_2^*, F_3^*, v^-\right) - \sum_{j=1}^{3} F_j^*(v^-)} \right] \Bigg\}.$$

Theorem 2: The SDF F_1^*, F_2^*, and F_3^* determine (uniquely) the DF F_2 of a SPS for $t \leq t^*$ by $F_2(t) = \Phi_{sp}(F_1^*, F_2^*, F_3^*, t)$, and the DF F_2 of a PSS for $t \leq t^*$ by $F_2(t) = \Phi_{ps}(F_1^*, F_2^*, F_3^*, t)$.

Note that Theorem 2 can be easily rewritten to obtain the relation of DF F_3 and the SDF. However, for the component X_1, we have a series system (or a parallel system). In this case, we can use the result given in Theorem 1.

Theorem 2 provides an important relation between the SDF and DF, for both the SPS and the PSS. Using this result, in the next section, we have developed the nonparametric Bayesian estimator for the DF of the system's components.

III. Bayesian Analysis

This section describes a Bayesian reliability approach to the SPS and the PSS. We have derived a nonparametric Bayesian estimator of the distribution function using the multivariate Dirichlet process [15], [17]. From Property 1, we have that the sub-distribution functions are related to the system distribution function by a sum. Considering that $F_1^*(t) + F_2^*(t) + F_3^*(t) + (1 - F(t)) = 1$, we have the restriction that these four quantities have a sum equal to 1, and that the set of possible points for $\{F_1^*(t), F_2^*(t), F_3^*(t), 1 - F(t)\}$ is the four-dimensional simplex, or $\{F_1^*(t), F_2^*(t), F_3^*(t)\}$ for the non-singular form. In this case, for a fixed t, we have that a natural prior choice is the Dirichlet distribution, and for any t, we have the Dirichlet multivariate process. The Dirichlet multivariate process can be viewed as a random distribution. Our interest is to develop a nonparametric estimator for the distribution function of the components in a SPS or in a PSS, and using the Dirichlet process, we have a complete distribution for the set $\{F_1^*(t), F_2^*(t), F_3^*(t)\}$. In this case, our parameters are the functions that we want to estimate, giving us a nonparametric framework. For a better understanding regarding the properties of the Dirichlet (univariate) process, see Ferguson [16]; for the multivariate Dirichlet processes, see Salinas-Torres *et al.* [17]; and for a simplified version, see [15].

Consider $\Omega = (0, \infty)$, the prior for $\mathbf{F}^* = (F_1^*, F_2^*, F_3^*)$, and the vector of the components' SDF is $\mathbf{F}^* \sim \mathcal{D}_k(\alpha_1, \alpha_2, \alpha_3)$. The induced prior for F_j^* is given by

$$F_j^*(t) \sim \text{Beta}\left(\alpha_j(0, t]; \alpha_j(t, \infty)\right), t > 0. \tag{6}$$

The following result gives the prior mean of the distribution function F_2 in terms of the prior mean of its associated cumulative reversed hazard rate (CRHR) $\mathcal{M}_2$ for the SPS, and cumulative hazard rate (CHR) Λ_2 for the PSS.

Lemma 1: Suppose that F_1, F_2, and F_3 have no common discontinuities. Under the prior (6), for $F_2^*(\cdot)$, the prior mean of the distribution function F_2, for each $t > 0$, is given by

$$F_{2,0}(t) = \mathrm{E}\left[F_2(t)\right] = \prod_t^{\infty} \left((1) - \mathrm{d}\mathcal{M}_{2,0}(s)\right),$$

for the SPS, and

$$F_{2,0}(t) = \mathrm{E}\left[F_2(t)\right] = 1 - \mathrm{E}\left[R_2(t)\right] = 1 - \prod_0^{t} \left((1) - \mathrm{d}\Lambda_{2,0}(s)\right),$$

for the PSS, where $\mathcal{M}_{2,0}(s) := \mathrm{E}[\mathcal{M}_2(s)]$ is the prior mean of $\mathcal{M}_2$ associated with the distribution function F_2 for the SPS, and $\Lambda_{2,0}(s) := \mathrm{E}[\Lambda_2(s)]$ is the prior mean of Λ_2 associated with the distribution function F_2 for the PSS.

The posterior distribution of $\mathbf{F}^*$ is an updated Dirichlet multivariate process where $\mathbf{F}^*(t)|n\mathbf{F}_n^*(t) \sim DM_3(\alpha_1(0, t] + nF_{1n}^*(t), \alpha_2(0, t] + nF_{2n}^*(t), \alpha_3(0, t] + nF_{3n}^*(t))$; see Salinas-Torres *et al.* [17]. The Bayesian estimators (posterior means) of F_j^* and F are given by

$$\widehat{F}_j^*(t) = p_\alpha \frac{\alpha_j(0, t]}{\sum_{\ell=1}^{3} \alpha_\ell(0, \infty)} + (1 - p_\alpha) F_{jn}^*(t), \tag{7}$$

where $p_\alpha = (\sum_{j=1}^{3} \alpha_j(0, \infty))/(n + \sum_{j=1}^{3} \alpha_j(0, \infty))$, and

$$\widehat{F}(t) = \sum_{j=1}^{3} \widehat{F}_j^*(t). \tag{8}$$

These Bayesian estimators are strongly s-consistent. For instance, using the Glivenko Cantelli Theorem (cf. Billingsley [18, pp. 275]), it can be shown that $\widehat{F}_j^*$ converges to F_j^* uniformly with probability 1.

If $\alpha_j(0, \infty) < \infty$, the Bayesian estimator of $\rho_j = \Pr(\delta = j)$ is given by

$$\widehat{\rho}_j = \lim_{t \uparrow \infty} \widehat{F}_j^*(t) = \frac{\alpha_j(0, \infty)}{n + \sum_{\ell=1}^{k} \alpha_\ell(0, \infty)} + \frac{\sum_{i=1}^{n} \mathbb{I}(\delta_i = j)}{n + \sum_{\ell=1}^{k} \alpha_\ell(0, \infty)}. \tag{9}$$

Let the $m(\leq n)$ distinct order statistics of T be $T_{(1)}^\bullet < \ldots < T_{(m)}^\bullet$. Set $N_i = \sum_{\ell=1}^{n} \mathbb{I}(T_\ell < T_{(i)}^\bullet)$, and $d_{ji} = \sum_{\ell=1}^{n} \mathbb{I}(T_\ell = T_{(i)}^\bullet, \delta_\ell = j)$, $i = 1, \ldots, m$. Define

$$I_s(t) = \exp\left\{ \frac{1}{\sum_{j=1}^{3} \alpha_j(0, \infty) + n} \int_0^t \frac{-\mathrm{d}\alpha_1(0, s]}{1 - \widehat{F}(s)} \right\}, \tag{10}$$

$$\Pi_s(t) = \prod_{i: T_{(i)}^\bullet \leq t} \frac{\sum_{j=1}^{3} \alpha_j(T_{(i)}^\bullet, \infty) + n - N_i - d_{1i}}{\sum_{j=1}^{3} \alpha_j\left(T_{(i)}^\bullet, \infty\right) + n - N_i}. \tag{11}$$

$$I_{sp}(t) = \exp\left\{ \frac{1}{\sum_{j=1}^{3} \alpha_j(0, \infty) + n} \int_t^\infty \frac{-\mathrm{d}\alpha_2(0, s]}{\widehat{F}(s) - \widehat{F}_1(s)} \right\}, \tag{12}$$

$$\Pi_{sp}(t) = \prod_{i: T_{(i)}^\bullet > t} \frac{\frac{\sum_{j=1}^{3} \alpha_j\left(0, T_{(i)}^\bullet\right] + N_i}{n + \sum_{j=1}^{3} \alpha_j(0, \infty)} - \widehat{F}_1(T_{(i)}^\bullet)}{\frac{\sum_{j=1}^{3} \alpha_j\left(0, T_{(i)}^\bullet\right] + N_i + d_{2i}}{n + \sum_{j=1}^{3} \alpha_j(0, \infty)} - \widehat{F}_1\left(T_{(i)}^\bullet\right)}. \tag{13}$$

$$I_p(t) = \exp\left\{ \frac{1}{\sum_{j=1}^{3} \alpha_j(0,\infty)+n} \int_t^\infty \frac{-\mathrm{d}\alpha_1(0,s]}{\widehat{F}(s)} \right\}, \quad (14)$$

$$\Pi_p(t) = \prod_{i:T^\bullet_{(i)}>t} \frac{\sum_{j=1}^{3} \alpha_j\left(0,T^\bullet_{(i)}\right]+N_i}{\sum_{j=1}^{3} \alpha_j\left(0,T^\bullet_{(i)}\right]+N_i+d_{1i}}. \quad (15)$$

$$I_{ps}(t) = \exp\left\{ \frac{1}{\sum_{j=1}^{3} \alpha_j(0,\infty)+n} \int_0^t \frac{-\mathrm{d}\alpha_2(0,s]}{\widehat{F}_1(s)-\widehat{F}(s)} \right\}, \quad (16)$$

and

$$\Pi_{ps}(t) = \prod_{i:T^\bullet_{(i)}\le t} \frac{\widehat{F}_1\left(T^\bullet_{(i)}\right) - \frac{\sum_{j=1}^{3} \alpha_j\left(0,T^\bullet_{(i)}\right]+N_i+d_{2i}}{n+\sum_{j=1}^{3}\alpha_j(0,\infty)}}{\widehat{F}_1\left(T^\bullet_{(i)}\right) - \frac{\sum_{j=1}^{3} \alpha_j\left(0,T^\bullet_{(i)}\right]+N_i}{n+\sum_{j=1}^{3}\alpha_j(0,\infty)}}. \quad (17)$$

The main result of this study is given in the following paragraph.

Theorem 3: Suppose that $\alpha_1(0,\cdot), \alpha_2(0,\cdot), \alpha_3(0,\cdot)$ are continuous on (t,∞), for each $t>0$, and F_1, F_2, and F_3 have no common discontinuities. Then, for $t \le T_{(n)}$, and the SPS, we have that

$$\begin{aligned}\widehat{F}_1(t) &= \mathrm{E}\left[F_1(t)|data\right]\\ &= \Phi_s\left(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t\right) = 1 - I_s(t)\Pi_s(t), \quad (18)\\ \widehat{F}_2(t) &= \mathrm{E}\left[F_2(t)|data\right]\\ &= \Phi_{sp}\left(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t\right) = I_{sp}(t)\Pi_{sp}(t); \quad (19)\end{aligned}$$

and, for the PSS,

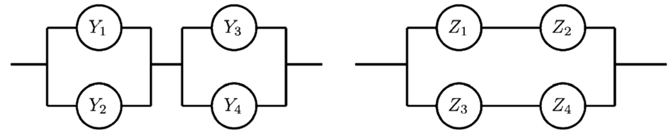
$$\begin{aligned}\widehat{F}_1(t) &= \mathrm{E}\left[F_1(t)|data\right]\\ &= \Phi_p\left(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t\right) = I_p(t)\Pi_p(t), \quad (20)\\ \widehat{F}_2(t) &= \mathrm{E}\left[F_2(t)|data\right]\\ &= \Phi_{ps}\left(\widehat{F}_1^*, \widehat{F}_2^*, \widehat{F}_3^*, t\right) = 1 - I_{ps}(t)\Pi_{ps}(t). \quad (21)\end{aligned}$$

$\widehat{F}_1(t)$, and $\widehat{F}_2(t)$ are the nonparametric estimators of $F_1(t)$, and $F_2(t)$, respectively, based on posterior means.

As in Theorem 2, it is straightforward to express the nonparametric estimator of $F_3(t)$. In the next section, we extend the estimators to a general case of $k \ge 4$.

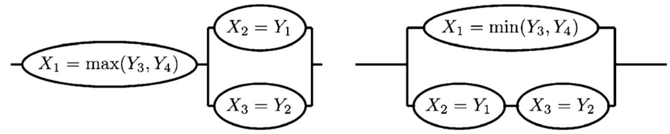
IV. Bayesian Estimator for $k \ge 4$

The extension of the nonparametric Bayesian estimator for the SPS and the PSS, given in Section III, is based on rewriting the system representation in a proper simplified version of the general case ($k \ge 4$) to the one given with $k = 3$, which has a solution given in Theorem 3. Considering the SPS and the PSS presented in Fig. 3, we specify how to rewrite the system representation and estimation of their components' reliability in the following.



(a) (b)

Fig. 3. (a) SPS, (b) PSS.



(a) (b)

Fig. 4. (a) SPS, (b) PSS.

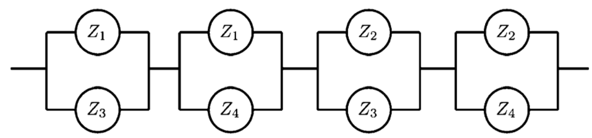


Fig. 5. The SPS representation of the PSS in Fig. 3(b).

We provided how to estimate Y_1 (for the SPS), and Z_1 (for the PSS), because the reliability estimation of the other components are straightforward once these two are given. The idea of the extension is to represent the systems in a simple version with three components (Figs. 1(a) and 2(a)). In this case, to estimate the reliability of Y_1, we use the SPS solution considering $X_1 = \max(Y_3, Y_4)$, $X_2 = Y_1$, and $X_3 = Y_2$ (Fig. 4(a)); and for the estimation of Z_1, we use the PSS solution considering $X_1 = \min(Y_3, Y_4)$, $X_2 = Y_1$, and $X_3 = Y_2$ (Fig. 4(b)). It must be noted that other more complex systems can also be considered, but the task is only to simplify the representation of the system as one of either the SPS or the PSS given in Figs. 1(a) and 2(a).

Furthermore, both the classes (SPS and PSS) are important so as to have a more general solution, because we have the restriction that two different components cannot have the same failure time, which in turn would result in different representations giving more options to the reliability estimation problem. Considering the PSS given in Fig. 3(b), we can write their SPS representation as that presented in Fig. 5. The component's reliability of the original PSS (Fig. 3(b)) can be estimated using the PSS result of Theorem 3, which has a simple solution. However, as the SPS representation (Fig. 5) has some components repeated, the SPS result of Theorem 3 is not applicable. Thus, the solutions for both the SPS and the PSS are important, and can be used in different situations.

V. Numerical Examples

This section presents two examples to demonstrate the estimation steps, and shows the quality of the Bayesian nonparametric estimator. The estimation steps for the PSS are very similar to those for the SPS, and for the sake of brevity, we have omitted them. The estimation steps for the SPS are as follows.

1) *Defining priors*: The prior measures $(\alpha_1, \alpha_2, \alpha_3)$ are prior guesses of the SDF (F_1^*, F_2^*, F_3^*), but it is not simple to elicit these measures. It is easier to elicit the priors for the DF (F_1, F_2, F_3), and use (4) for the SPS to evaluate the prior measures (for the PSS we can use (5)). In our case, we chose the exponential distribution (with mean 1) as the prior guess for each of the three components' DF. By evaluating the prior measures using (4), we have $\alpha_1(0,v] = (e^{-3v} - 3e^{-2v} + 2)/3$, and $\alpha_2(0,v] = \alpha_3(0,v] = (2e^{-3v} - 3e^{-2v} + 1)/6$. Note that this prior is not very informative because the measure of the whole parameter space is only one $(\alpha_1(\Omega) + \alpha_2(\Omega) + \alpha_3(\Omega) = 1)$. Also, we have that $\mathrm{d}\alpha_1(0,v] = 2e^{-2v} - e^{-3v}\mathrm{d}v$, and $\mathrm{d}\alpha_2(0,v] = \mathrm{d}\alpha_3(0,v] = e^{-2v} - e^{-3v}\mathrm{d}v$.
2) *Obtaining Posteriors*: The posterior processes for the SDF functions are $DM_3((e^{-3t} - 3e^{-2t} + 2)/3 + nF_{1n}^*(t), (2e^{-3t} - 3e^{-2t} + 1)/6 + nF_{2n}^*(t), (2e^{-3t} - 3e^{-2t} + 1)/6 + nF_{3n}^*(t))$; and from (7), we have

$$\widehat{F}_1^*(t) = \frac{nF_{1n}^*(t) + (e^{-3t} - 3e^{-2t} + 2)/3}{n+1},$$
$$\widehat{F}_2^*(t) = \frac{nF_{2n}^*(t) + (2e^{-3t} - 3e^{-2t} + 1)/6}{n+1},$$
$$\widehat{F}_3^*(t) = \frac{nF_{3n}^*(t) + (2e^{-3t} - 3e^{-2t} + 1)/6}{n+1},$$

which are the estimators of SDF.
3) *Computing system's reliability*: (8) provides the estimator of the system distribution function. For the prior defined earlier, we have

$$\widehat{F}(t) = \frac{nF_n^*(t) + e^{-3t} - 2e^{-2t} + 1}{n+1}.$$

4) *Computing components' reliabilities*: Theorem 3 gives the estimators of the components' DF. Using (18), we obtain the estimate for component 1 DF; and from (19), we obtain the estimate for component 2 DF. For component 3 DF, we substitute $\mathrm{d}\alpha_2(0,v]$ by $\mathrm{d}\alpha_3(0,v]$ (in the integral part I_{sp}), and d_{2i} by d_{3i} (in the product part Π_{sp}) in (19). Also, the integral part of the estimator can be solved by using a numerical procedure, such as the Simpson's rule. For more details and other numerical integration methods, see Davis and Rabinowitz [19].

Example 1: We obtained 100 observations of four simulated processes, where all the components had gamma distributions, and the first component (Y_1) had a mean of 4 and a standard deviation (SD) of 2.83, the second component (Y_2) had a mean of 6 and a SD of 4.9, the third component (Y_3) had a mean of 8 and a SD of 5.67, and the fourth component (Y_4) had a mean of 3 and a SD of 2.45. Let us consider the SPS presented in Fig. 3(a). The Bayesian estimators are based on 100 observations of (T, δ). The simulated values are listed in the Appendix.

To estimate components 1–4, we rewrote the representation of the system as follows. For component 1, we considered that $X_1 = \max(Y_3, Y_4)$, $X_2 = Y_1$, and $X_3 = Y_2$; then $\widehat{F}_{Y_1}(t) = \widehat{F}_2(t)$, where $\widehat{F}_{Y_1}$ is the DF estimate of component 1, and $\widehat{F}_2$ is the proposed estimator for the SPS (19). In a similar way, for component 2, we considered $X_1 = \max(Y_3, Y_4)$, $X_2 = Y_2$, and $X_3 = Y_1$; for component 3, we considered $X_1 = \max(Y_1, Y_2)$, $X_2 = Y_3$, and $X_3 = Y_4$; and for component 4, we considered $X_1 = \max(Y_1, Y_2)$, $X_2 = Y_4$, and $X_3 = Y_3$ (see Section IV). We found that the proportions of censored data for components 1–4 are 77%, 64%, 73%, and 86%, respectively.

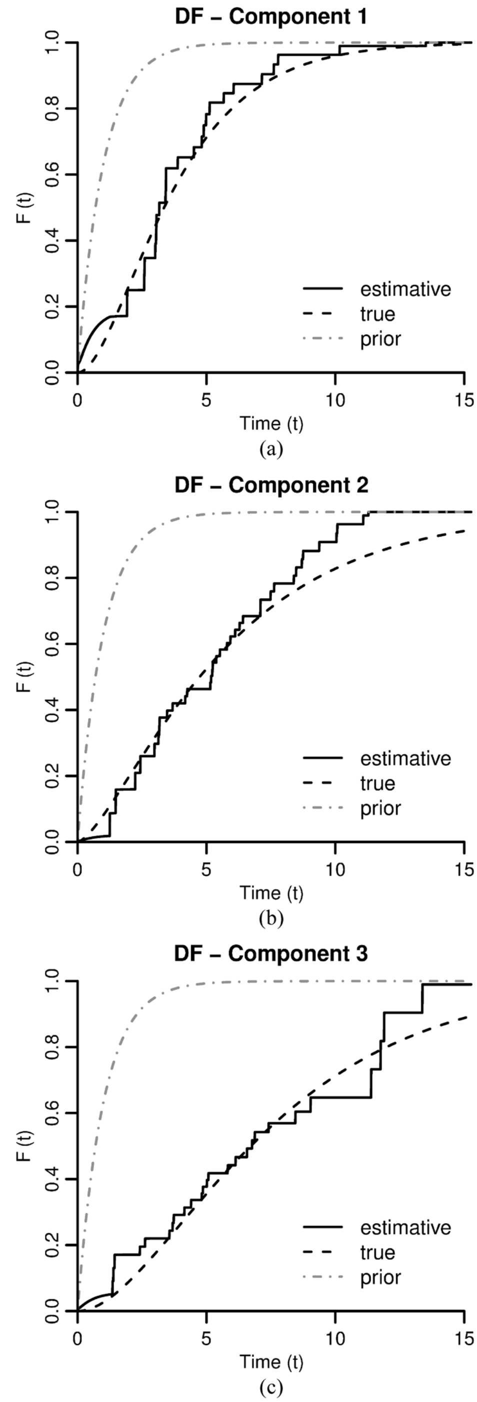


Fig. 6. Estimates for the Example 1.

Fig. 6(a) –6(e) present the estimates of the five distribution functions associated with components 1–4, and the system. In all plots, the *true* distribution functions (dashed lines) and the prior mean (dashed-dot line) are also illustrated. The conditional reliabilities of the components relative to the system are $\widehat{\rho}_1 \cong 0.2294$, $\widehat{\rho}_2 \cong 0.3581$, $\widehat{\rho}_3 \cong 0.2690$, and $\widehat{\rho}_4 \cong 0.1403$.

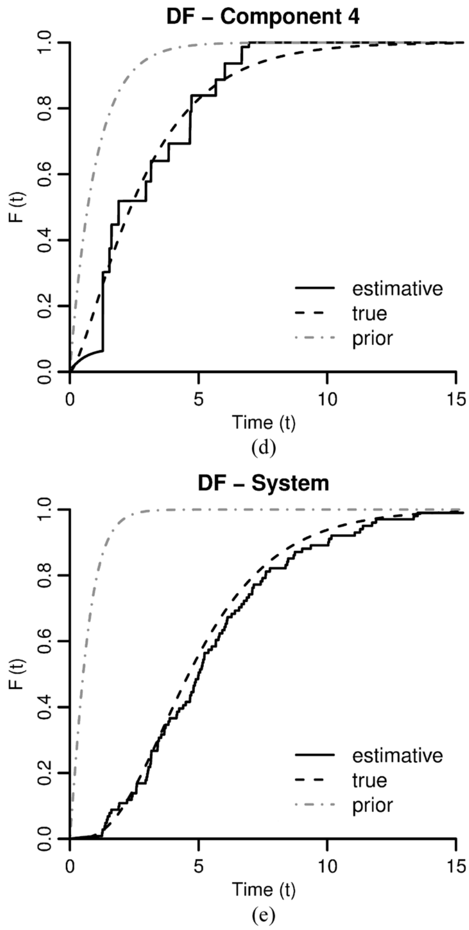


Fig. 6. *(Continued.)* Estimates for the Example 1.

TABLE I
SUMMARY STATISTICS FOR MAE OF 1000 s-INDEPENDENT COPIES OF A SYSTEM WITH $n = 30$ units

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
$\widehat{F}_{Y_1}$	0.0330	0.0731	0.1004	0.1119	0.1385	0.3485
$\widehat{F}_{Y_2}$	0.0321	0.0802	0.1162	0.1326	0.1706	0.4537
$\widehat{F}_{Y_3}$	0.0337	0.0988	0.1505	0.1715	0.2264	0.5923
$\widehat{F}_{Y_4}$	0.0439	0.0903	0.1218	0.1350	0.1744	0.3327
$\widehat{F}_{Y}$	0.0182	0.0415	0.0558	0.0630	0.0768	0.2095

To better understand the performance of the estimator, we did a simulation study of the system, shown in Example 1. For each of the three sample sizes ($n = 30$, 100, and 1000), we generated 1000 s-independent copies. Then, for each of these 1000 different copies (data sets), we evaluated the mean absolute error (MAE) from the estimator to the true distribution. Tables I through III present the summary statistics obtained. As expected, if the sample size is large ($n = 1000$), then the estimator is better (that is, it has the smallest summary statistics for MAE). With the small sample size ($n = 30$), we obtained the worst (largest) summary statistics. For instance, the overall worst mean of MAE was achieved at the component Y_3 (around 0.17).

TABLE II
SUMMARY STATISTICS FOR MAE OF 1000 s-INDEPENDENT COPIES OF A SYSTEM WITH $n = 100$ units

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
$\widehat{F}_{Y_1}$	0.0172	0.0402	0.0523	0.0591	0.0714	0.1807
$\widehat{F}_{Y_2}$	0.0149	0.0425	0.0613	0.0712	0.0923	0.2408
$\widehat{F}_{Y_3}$	0.0194	0.0529	0.0786	0.0888	0.1155	0.3263
$\widehat{F}_{Y_4}$	0.0223	0.0519	0.0670	0.0761	0.0929	0.2586
$\widehat{F}_{Y}$	0.0100	0.0217	0.0283	0.0316	0.0388	0.1024

TABLE III
SUMMARY STATISTICS FOR MAE OF 1000 s-INDEPENDENT COPIES OF A SYSTEM WITH $n = 1000$ units

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
$\widehat{F}_{Y_1}$	0.0050	0.0128	0.0166	0.0184	0.0221	0.0614
$\widehat{F}_{Y_2}$	0.0056	0.0136	0.0187	0.0220	0.0280	0.0708
$\widehat{F}_{Y_3}$	0.0060	0.0157	0.0247	0.0277	0.0367	0.0933
$\widehat{F}_{Y_4}$	0.0074	0.0161	0.0210	0.0232	0.0282	0.0663
$\widehat{F}_{Y}$	0.0034	0.0069	0.0091	0.0102	0.0125	0.0307

Example 2: In this example, we considered a PSS with four components; one of them had the distribution function of a mixture of an exponential distribution and a discrete distribution, with positive probability to fail at times 1 and 3, and is given by

$$F(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.6(1 - e^{-t/4}), & \text{if } 0 < t \leq 1, \\ 0.6(1 - e^{-t/4}) + 0.25, & \text{if } 1 < t \leq 3, \\ 0.6(1 - e^{-t/4}) + 0.4, & \text{if } t > 3. \end{cases} \tag{22}$$

We obtained 100 observations of four simulated processes, where the first component (Z_1) had a gamma distribution with a mean of 4 and a SD of 2.83, the second component (Z_2) had a Weibull distribution with a mean of 4.51 and a SD of 3.06, the third component (Z_3) had a mixture of an exponential and a discrete distribution (22) with a mean of 3.1 and a SD of 3.34, and the fourth component (Z_4) had a log-normal distribution with a mean of 4.59 and a SD of 2.45. Let us now consider the PSS presented in Fig. 3(b). The Bayesian estimators are based on 100 observations of (T, δ). The simulated values are listed in the Appendix.

To estimate the parameters for components 1 through 4, we rewrote the representation of the system as follows. For component 1, we considered that $X_1 = \min(Z_3, Z_4)$, $X_2 = Z_1$, and $X_3 = Z_2$; then, $\widehat{F}_{Z_1}(t) = \widehat{F}_2(t)$, where $\widehat{F}_{Z_1}$ is the DF estimate of component 1, and $\widehat{F}_2$ is the proposed estimator for the PSS (21). In a similar way, for component 2, we considered $X_1 = \min(Z_3, Z_4)$, $X_2 = Z_2$, and $X_3 = Z_1$; for component 3, we considered $X_1 = \min(Z_1, Z_2)$, $X_2 = Z_3$, and $X_3 = Z_4$; and for component 4, we considered $X_1 = \min(Z_1, Z_2)$, $X_2 = Z_4$, and $X_3 = Z_3$ (see Section IV). We found that the proportion of the censored data for components 1 through 3 is 71%, and that for component 4 is 87%.

Figs. 7(a) through 7(e) present the estimates of the five distribution functions: components 1 through 4, and the system. In all the plots, the *true* distribution functions (dashed lines) and

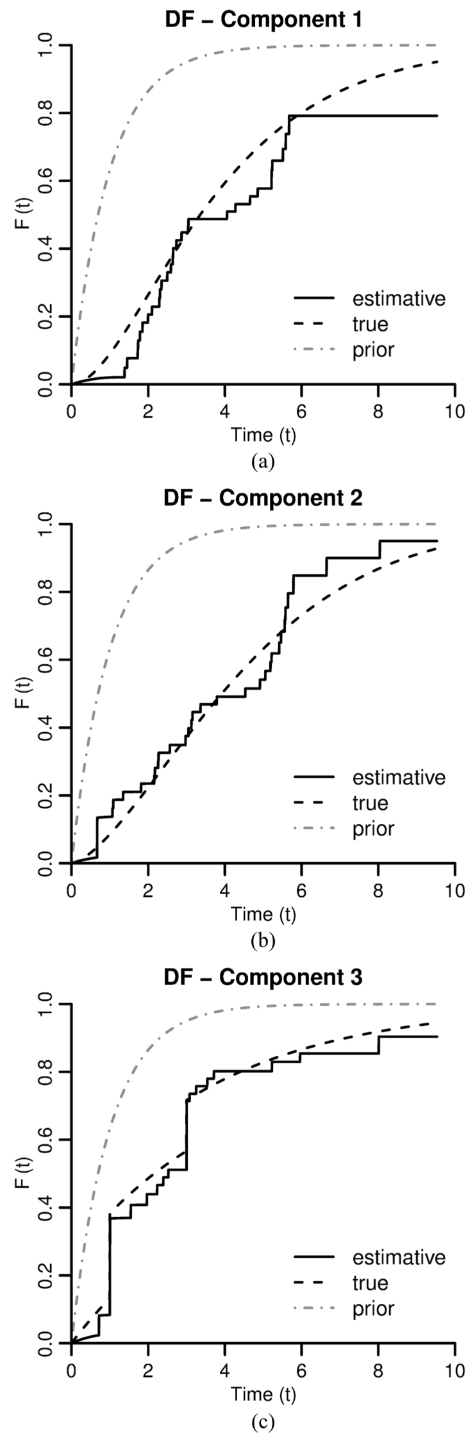


Fig. 7. Estimates for the Example 2.

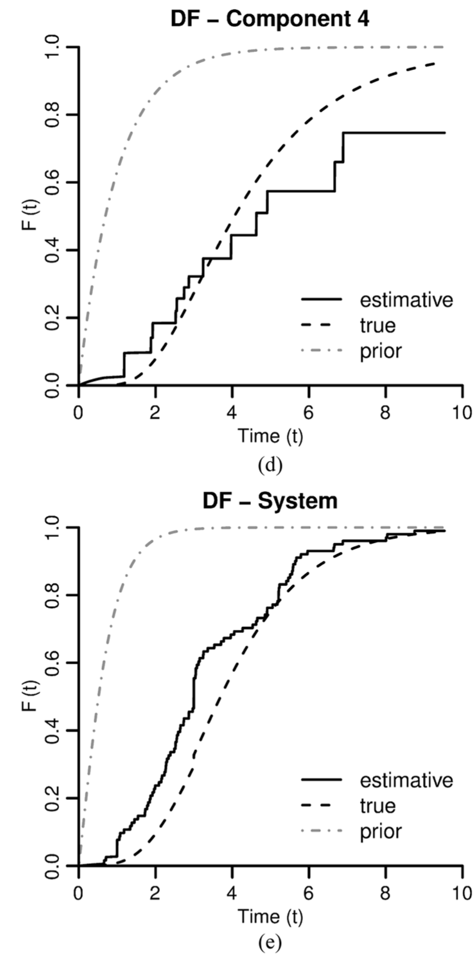


Fig. 7. *(Continued.)* Estimates for the Example 2.

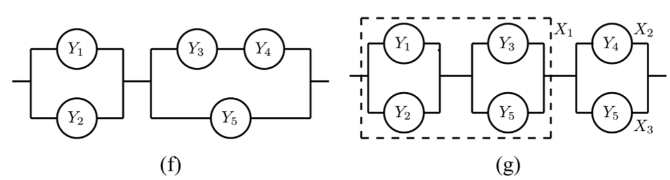


Fig. 7. *(Continued.)* (f) A more complex system; (g) SPS representation of system in (f).

the prior mean (dashed-dot line) are also illustrated. The conditional reliabilities of the components relative to the system are $\widehat{\rho}_1 \cong 0.2887$, $\widehat{\rho}_2 \cong 0.2887$, $\widehat{\rho}_3 \cong 0.2887$, and $\widehat{\rho}_4 \cong 0.1303$.

VI. Concluding Remarks, and Areas for Further Research

Salinas-Torres *et al.* [13], and Polpo and Sinha [14] carried out the nonparametric Bayesian estimation of components' reliability in a series system, while Polpo and Pereira [15] presented the estimation for the parallel system. In the present study, we have extended both these earlier works to a more general problem of estimating components' reliabilities in a SPS or a PSS. The product integral was necessary to prove the Theorem 3, which provides the mean posterior estimator for a component's distribution function. The proposed estimator can accommodate both continuous and discrete failure times (see Example 2), and is s-consistent. In this case, the user of our proposed estimator does not have to be worried if the unknown reliability function to be estimated is continuous, discrete, or a mixing of both because the estimator can accommodate all these cases, giving a very general solution.

The estimation of more complex structures than those presented in Fig. 3(a) or Fig. 3(b) can be done by considering sub-systems, and some adaptations on the initial problem. For example, if our interest is the estimation of component Y_5 in the system given in Fig. 8(a), then we can build a new system by taking $X_1 = \max(Y_1, Y_2)$, $X_2 = Y_5$, and $X_3 = \min(Y_3, Y_4)$, where X_1, X_2, X_3 are the components of the system given in Fig. 1(a), and we can just estimate the system (1(a)) as earlier. On the other hand, if our interest is in the estimation of component Y_4, and we have $X_1 = \min(\max(Y_1, Y_2), \max(Y_3, Y_5))$, $X_2 = Y_4$, and $X_3 = Y_5$ (Fig. 8(b)), then, in this case, the jump point sets of X_1 and X_3 cannot be disjoint, and this result violates the estimator assumptions. Hence, we can estimate Y_1, Y_2, and Y_5, and cannot estimate Y_3, or Y_4. The estimator proposed in the present study opens new possibilities in reliability estimation; however, it is not the final solution for coherent systems. As future research, one can think about how to solve the question of the assumption of a disjoint jump point set, where it will be possible to estimate the reliability of more complex systems, such as the bridge system.

APPENDIX

Proof of Theorem 1: For the first part, see Peterson [11, Theorem 2.1], and for the second part see Polpo and Pereira [15, Theorem 2]. ■

Proof of Theorem 2: For the SPS, we have that, from the reversed hazard rate (RHR) (see Polpo and Pereira [15], Block *et al.* [20], and Li and Zuo [21]),

$$F_2(t) = \exp\left\{-\oint_t^\infty \mu_c^{F_2}(v)\mathrm{d}v\right\} \prod_{v>t}^{F_2} \left[1 - \mu_d^{F_2}(v)\right]. \quad (23)$$

We can write the integration as

$$\begin{aligned}
\oint_t^\infty \mu_c^{F_2}(v)\mathrm{d}v &= \oint_t^\infty \frac{\mathrm{d}F_2(v)}{F_2(v)} \\
&= \oint_t^\infty \frac{[1 - F_1(v)]\, F_3(v)\mathrm{d}F_2(v)}{[1 - F_1(v)]\, F_3(v)F_2(v)} \\
&= \oint_t^\infty \frac{\mathrm{d}F_2^*(v)}{F(v) - F_1(v)} \\
&= \oint_t^\infty \frac{\mathrm{d}F_2^*(v)}{\sum_{j=1}^3 F_j^*(v) - \Phi_s\left(F_1^*, F_2^*, F_3^*, v\right)}.
\end{aligned}$$

From (1), $F(v) = \sum_{j=1}^3 F_j^*(v)$; from (2), $F(t) - F_1(v) = [1 - F_1(v)]F_3(v)F_2(v)$; from (4), $\mathrm{d}F_2^*(v) = [1 - F_1(v)]F_3(v)\mathrm{d}F_2(v)$; and from Theorem 1, $F_1(v) = \Phi_s(F_1^*, F_2^*, F_3^*, v)$. Also, the product becomes

$$\begin{aligned}
\prod_{v>t}^{F_2}\left[1 - \mu_d^{F_2}(v)\right]
&= \prod_{v>t}^{F_2}\left[\frac{F_2(v^-)}{F_2(v^+)}\right] \\
&= \prod_{v>t}^{F_2}\left[\frac{[1 - F_1(v^-)]\, F_3(v^-)F_2(v^-)}{[1 - F_1(v^+)]\, F_3(v^+)F_2(v^+)}\right] \\
&= \prod_{v>t}^{F_2^*}\left[\frac{F(v^-) - F_1(v^-)}{F(v^+) - F_1(v^+)}\right] \\
&= \prod_{v>t}^{F_2^*}\left[\frac{\sum_{j=1}^3 F_j^*(v^-) - \Phi_s\left(F_1^*, F_2^*, F_3^*, v^-\right)}{\sum_{j=1}^3 F_j^*(v^+) - \Phi_s\left(F_1^*, F_2^*, F_3^*, v^+\right)}\right].
\end{aligned}$$

Note that, from Property 3, $F_1(v^-) = F_1(v^+)$ and $F_3(v^-) = F_3(v^+)$, and the last equality holds. Because F_2 is positive and increasing, $F_2(t_\ell) = 0$ implies $F_2(t) = 0$ for $t < t_\ell$, and $F_2(t_u) = 1$ implies $F_2(t) = 1$ for $t > t_u$.

For the PSS, we have that, from the hazard rate (HR) (see Peterson [11]),

$$F_2(t) = \exp\left\{-\oint_0^t \lambda_c^{F_2}(v)\mathrm{d}v\right\} \prod_{v\le t}^{F_2}\left[1 - \lambda_d^{F_2}(v)\right]. \quad (24)$$

We can write the integration as

$$\begin{aligned}
\oint_0^t \lambda_c^{F_2}(v)\mathrm{d}v &= \oint_0^t \frac{-\mathrm{d}\left[1 - F_2(v)\right]}{1 - F_2(v)} \\
&= \oint_0^t \frac{-F_1(v)\left[1 - F_3(v)\right]\mathrm{d}\left[1 - F_2(v)\right]}{F_1(v)\left[1 - F_3(v)\right]\left[1 - F_2(v)\right]} \\
&= \oint_0^t \frac{\mathrm{d}F_2^*(v)}{F_1(v) - F(v)} \\
&= \oint_0^t \frac{\mathrm{d}F_2^*(v)}{\Phi_p\left(F_1^*, F_2^*, F_3^*, v\right) - \sum_{j=1}^3 F_j^*(v)}.
\end{aligned}$$

From (1), $F(v) = \sum_{j=1}^3 F_j^*(v)$; from (3), $F_1(v) - F(v) = F_1(v)[1 - F_3(v)][1 - F_2(v)]$; from (5), $\mathrm{d}F_2^*(v) = -F_1(v)[1 - F_3(v)]\mathrm{d}[1 - F_2(v)]$; and from Theorem 1, $F_1(v) = \Phi_p(F_1^*, F_2^*, F_3^*, v)$. Also, the product becomes

$$\begin{aligned}
\prod_{v\le t}^{F_2}\left[1 - \lambda_d^{F_2}(v)\right]
&= \prod_{v\le t}^{F_2}\left[\frac{1 - F_2(v^+)}{1 - F_2(v^-)}\right] \\
&= \prod_{v\le t}^{F_2}\left[\frac{F_1(v^+)\left[1 - F_3(v^+)\right]\left[1 - F_2(v^+)\right]}{F_1(v^-)\left[1 - F_3(v^-)\right]\left[1 - F_2(v^-)\right]}\right] \\
&= \prod_{v\le t}^{F_2^*}\left[\frac{F_1(v^+) - F(v^+)}{F_1(v^-) - F(v^-)}\right] \\
&= \prod_{v\le t}^{F_2^*}\left[\frac{\Phi_p\left(F_1^*, F_2^*, F_3^*, v^+\right) - \sum_{j=1}^3 F_j^*(v^+)}{\Phi_s\left(F_1^*, F_2^*, F_3^*, v^-\right) - \sum_{j=1}^3 F_j^*(v^-)}\right].
\end{aligned}$$

Again, from Property 3, $F_1(v^-) = F_1(v^+)$ and $F_3(v^-) = F_3(v^+)$, and the last equality holds. Because F_2 is positive and increasing, $F_2(t_\ell) = 0$ implies $F_2(t) = 0$ for $t < t_\ell$, and $F_2(t_u) = 1$ implies $F_2(t) = 1$ for $t > t_u$. ■

TABLE IV
SIMULATED SAMPLE OF EXAMPLE 1

T	δ	T	δ	T	δ	T	δ	T	δ
1.25	2	1.27	4	1.35	3	1.38	3	1.43	3
1.47	2	1.54	4	1.61	4	1.89	4	1.92	1
2.22	2	2.40	3	2.43	2	2.58	1	2.59	1
2.60	3	2.94	4	2.98	2	3.01	1	3.03	1
3.05	1	3.12	2	3.14	4	3.15	2	3.16	1
3.17	2	3.40	1	3.41	1	3.42	1	3.46	2
3.55	3	3.67	3	3.68	2	3.72	3	3.83	4
3.88	1	4.14	3	4.17	2	4.24	2	4.40	3
4.50	1	4.66	4	4.67	4	4.72	4	4.80	1
4.81	3	4.85	3	4.89	1	4.98	1	5.00	3
5.06	3	5.12	1	5.15	2	5.18	2	5.22	2
5.23	2	5.38	2	5.51	2	5.67	4	5.68	1
5.78	2	5.81	3	5.92	2	6.01	4	6.05	1
6.10	2	6.12	3	6.28	2	6.41	2	6.57	3
6.67	4	6.76	3	6.87	3	6.95	4	7.08	2
7.09	2	7.14	1	7.41	3	7.48	2	7.60	1
7.62	2	7.77	1	8.38	2	8.44	3	8.48	2
8.69	2	8.74	2	9.02	3	9.36	2	10.04	2
10.07	2	10.16	1	11.07	2	11.27	2	11.38	3
11.74	3	11.89	3	13.36	3	13.51	1	15.27	3

TABLE V
SIMULATED SAMPLE OF EXAMPLE 2

T	δ	T	δ	T	δ	T	δ	T	δ
0.66	2	0.71	3	1.00	3	1.00	3	1.00	3
1.00	3	1.00	3	1.06	2	1.08	2	1.18	4
1.34	2	1.38	1	1.45	1	1.54	3	1.72	1
1.73	1	1.78	1	1.81	2	1.85	1	1.87	4
1.92	4	1.96	3	1.99	1	2.09	1	2.15	2
2.17	2	2.23	3	2.26	2	2.27	2	2.28	1
2.30	1	2.34	1	2.39	3	2.49	1	2.52	3
2.53	4	2.55	4	2.56	2	2.59	1	2.64	1
2.65	1	2.73	1	2.74	4	2.87	1	2.88	4
2.97	2	3.00	3	3.00	3	3.00	3	3.00	3
3.00	3	3.00	3	3.00	3	3.00	3	3.00	3
3.04	1	3.05	1	3.06	2	3.08	3	3.12	2
3.15	2	3.24	4	3.25	3	3.36	2	3.53	3
3.71	3	3.78	2	3.96	4	4.05	1	4.27	1
4.53	2	4.62	4	4.65	1	4.85	1	4.91	2
4.92	4	5.05	2	5.18	2	5.21	2	5.22	1
5.23	1	5.24	3	5.25	1	5.41	2	5.46	2
5.51	1	5.56	2	5.58	2	5.59	1	5.64	2
5.67	1	5.79	2	5.96	3	6.64	2	6.66	4
6.88	4	8.01	3	8.04	2	8.75	3	9.53	3

Proof of Lemma 1: See Salinas-Torres *et al.* [13]. ■

Proof of Theorem 3: SPS part: for the proof of (18), see Polpo and Sinha [14]. Replacing the Bayesian estimates of F_1^*, F_2^*, and F_3^* in (19), we have

$$\widehat{F}_2(t)=\left\{\exp\left[\oint_t^{\infty}\frac{-\mathrm{d}\widehat{F}_2^*(v)}{\sum_{j=1}^{3}\widehat{F}_j^*(v)-\Phi_s\left(\widehat{F}_1^*,\widehat{F}_2^*,\widehat{F}_3^*,v\right)}\right]\right.$$
$$\left.\times\prod_{v>t}^{\widehat{F}_2^*}\left[\frac{\sum_{j=1}^{3}\widehat{F}_j^*(v^-)-\Phi_s\left(\widehat{F}_1^*,\widehat{F}_2^*,\widehat{F}_3^*,v^-\right)}{\sum_{j=1}^{3}\widehat{F}_j^*(v^+)-\Phi_s\left(\widehat{F}_1^*,\widehat{F}_2^*,\widehat{F}_3^*,v^+\right)}\right]\right\},\ t\leq T_{(n)}. \tag{25}$$

Note that $\mathrm{d}\widehat{F}_2^*(v)=\mathrm{d}\alpha_j(0,v]/\left(n+\sum_{\ell=1}^{3}\alpha_\ell(0,\infty)\right)$, and from (1) and (18) the first term in (25) becomes $I_{ps}(t)$, and the second factor in (25) is

$$\prod_{v>t}^{\widehat{F}_2^*}\frac{\dfrac{\sum_{j=1}^{3}\alpha_j(0,v]+\sum_{i=1}^{n}\mathbb{I}(T_i\leq v^-)}{n+\sum_{j=1}^{3}\alpha_j(0,\infty)}-\widehat{F}_1(v)}{\dfrac{\sum_{j=1}^{3}\alpha_j(0,v]+\sum_{i=1}^{n}\mathbb{I}(T_i\leq v^+)}{n+\sum_{j=1}^{3}\alpha_j(0,\infty)}-\widehat{F}_1(v)}=\Pi_{sp}(t).$$

On the other hand, proceeding as in Lemma 1,

$$\widehat{F}_2(t)=\mathrm{E}\left[F_2(t)|data\right]=\mathop{\pi}_{t}^{\infty}\left(1-\mathrm{d}\widehat{\mathcal{M}}_2(s)\right), \tag{26}$$

where $\mathrm{d}\widehat{\mathcal{M}}_2(s)=(c_3\mathrm{d}F_{2,0}^*(s)+n\mathrm{d}\widehat{F}_2^*(s))/(c_3F_0(s)+n\widehat{F}(s))$, and $c_3=\sum_{j=1}^{3}\alpha_j(0,\infty)$. With simple algebraic manipulations, we obtain (25) from (26).

PSS part: for the proof of (20), see Polpo and Pereira [15]. Replacing the Bayesian estimates of F_1^*, F_2^*, and F_3^* in (21), we have

$$\widehat{F}_2(t)=1-\left\{\exp\left[\oint_0^{t}\frac{-\mathrm{d}\widehat{F}_2^*(v)}{\Phi_p\left(\widehat{F}_1^*,\widehat{F}_2^*,\widehat{F}_3^*,v\right)-\sum_{j=1}^{3}\widehat{F}_j^*(v)}\right]\right.$$
$$\left.\times\prod_{v\leq t}^{\widehat{F}_2^*}\left[\frac{\Phi_p\left(\widehat{F}_1^*,\widehat{F}_2^*,\widehat{F}_3^*,v^+\right)-\sum_{j=1}^{3}\widehat{F}_j^*(v^+)}{\Phi_p\left(\widehat{F}_1^*,\widehat{F}_2^*,\widehat{F}_3^*,v^-\right)-\sum_{j=1}^{3}\widehat{F}_j^*(v^-)}\right]\right\},\ t\leq T_{(n)}. \tag{27}$$

Note that $\mathrm{d}\widehat{F}_2^*(v)=(\mathrm{d}\alpha_j(0,v])/(n+\sum_{\ell=1}^{3}\alpha_\ell(0,\infty))$, and from (1) and (20) the first term in (27) becomes $I_{ps}(t)$, and the second factor in (27) is

$$\prod_{v\leq t}^{\widehat{F}_2^*}\frac{\widehat{F}_1(v)-\dfrac{\sum_{j=1}^{3}\alpha_j(0,v]+\sum_{i=1}^{n}\mathbb{I}(T_i\leq v^+)}{n+\sum_{j=1}^{3}\alpha_j(0,\infty)}}{\widehat{F}_1(v)-\dfrac{\sum_{j=1}^{3}\alpha_j(0,v]+\sum_{i=1}^{n}\mathbb{I}(T_i\leq v^-)}{n+\sum_{j=1}^{3}\alpha_j(0,\infty)}}=\Pi_{ps}(t).$$

On the other hand, proceeding as in Lemma 1,

$$\widehat{F}_2(t)=\mathrm{E}\left[F_2(t)|data\right]=1-\mathop{\pi}_{0}^{t}\left(1-\mathrm{d}\widehat{\Lambda}_2(s)\right), \tag{28}$$

where $\mathrm{d}\widehat{\Lambda}_2(s)=(c_3\mathrm{d}F_{2,0}^*(s)+n\mathrm{d}\widehat{F}_2^*(s))/(c_3F_0(s)+n\widehat{F}(s))$, and $c_3=\sum_{j=1}^{3}\alpha_j(0,\infty)$. With simple algebraic manipulations, we obtain (27) from (28). ■

The simulated samples of $(T,\delta)_i$, $i=1,\ldots,100$, used in the examples, are described in Tables IV and V.

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Adriano Polpo received his BS degree in Statistics at the State University of Campinas (Brazil) in 2001, and the Ph.D. degree in Statistics at the University of São Paulo (Brazil) in 2005. He also did a post-doctoral at Florida State University (USA) in 2011–2012. Dr. Polpo is presently an Assistant Professor at the Federal University of São Carlos (Brazil), and president elected of ISBrA (2013–2014)—the Brazilian society of Bayesian statistics. His current interests include reliability and survival analysis, in both a theoretical and practical sense.

Debajyoti Sinha received his BS degree in Statistics at the Indian Statistical Institute (India) in 1986, and the Ph.D. degree in Statistics at the University of Rochester (USA) in 1993. He was Director of Biostatistics Division of Department of Biostatistics, Bioinformatic and Epidemiology, Medical University of South Carolina. Dr. Sinha is presently the Hobbs Endowed Professor of Statistics at the Department of Statistics, Florida State University, and it is an Elected fellow of American Statistical Association (2006). He has authored and co-authored more than 100 papers and 2 books on statistical theory, methods and applications.

Carlos A. de B. Pereira received his BS degree in Statistics at the National School of Statistical Science (Brazil) in 1968, MS degree in Statistics at the University of São Paulo (Brazil) in 1971, MS degree in Statistics at the Florida State University (USA) in 1978, and the Ph.D. degree in Statistics at the Florida State University (USA) in 1980. He was President of the Brazilian Statistical Society (1998–1990), Director of the Institute of Mathematic and Statistics, University of São Paulo (1994–1998), and Director of the Bioinformatic Scientific Center, University of São Paulo (2006–2009). Dr. Pereira is presently a Full Professor and Head of the Department of Statistics, University of São Paulo, and he was an Elected member of the International Statistical Institute. He has authored and co-authored more than 150 papers and 4 books.