Brazilian Journal of Probability and Statistics
2009, Vol. 23, No. 2, 216–226
DOI: 10.1214/08-BJPS011

A note on extendibility and predictivistic inference in finite populations

Pilar L. Iglesias[a], Rosangela H. Loschi[b], Carlos A. B. Pereira[c] and Sergio Wechsler[c]

[a]*Pontificia Universidad Católica de Chile*
[b]*Universidade Federal de Minas Gerais*
[c]*Universidade de São Paulo*

Abstract. The usual finite population model—where information provided by a subset of units is used to reduce uncertainty about functions of the complete population list of values—is explored from a predictivistic point of view. Under this approach, only operationally meaningful quantities (operational parameters) are considered and therefore no superpopulation parameters are involved. This paper addresses the estimation of both population total and maximum based on uniformity and/or exchangeability judgments on finite sequences of random variables. A central point of this paper is that there are contexts in which the superpopulation approach cannot be employed in inferential problems in finite populations. There are circumstances in which the prior distributions for the operational parameters cannot be obtained from any superpopulation model. Conditions for the extendibility to infinite populations are also established for some models, as this approach may ease the inferential problem.

1 Introduction

In this paper, a finite population of known size N is understood as a set of clearly labeled units, $P = \{1, 2, \dots, N\}$. To each unit there is associated a real-valued number or vector, so that we have the set of unknowns $X = (X_1, X_2, \dots, X_N) \in \mathcal{X}^N$. The sample space set $\mathcal{X}$ is known. Each X_i becomes known after unit i is (possibly) inspected. Statistical inference on some operational parameter T is to be made incorporating the information provided by a sample y. The *parameter* T is a function $T = T(X_1, X_2, \dots, X_N)$ and a *sample* of $n < N$ units *becomes available*. The sample is composed of the labels of inspected units and their values x_i. In other words, a sample y is the pair of collections $y = (s, (x_i : i \in s))$, where $s \subset P$. Associated to each set of labels $s = \{i_1, i_2, \dots, i_n\}$, there is the operator S defined by $S(X_1, X_2, \dots, X_N) = (X_{i_1}, X_{i_2}, \dots, X_{i_N})$. For simplicity it is assumed that $s = \{1, 2, \dots, n\}$ which is not a restriction under the assumption of an exchangeable prior distribution for X.

Key words and phrases. De Finetti-style theorems, exchangeability, extendibility, operational parameters, superpopulation models.
Received February 2007; accepted May 2008.

A main point of this note is that such a population is not only finite in its size, but also closed, that is, it should be protected against the introduction of models possessing unobservable (and therefore lacking operational meaning) quantities. The sole existing randomness is due to the ignorance of (some) of the x_i's, and the probability statements ought to refer exclusively to X. This approach to inference is known as operational Bayesian or predictivistic approach and was introduced by de Finetti (1937, 1975). See Daboni and Wedlin (1982) for a very detailed work on predictivistic methodology of Statistical Inference. See also Mendel (1994) and Wechsler (1993) for further discussion.

It is well known that most approaches do not keep the population closed:

(a) Frequentist, Cochran-style sampling models (Cochran (1977)): First of all, this approach is non-Bayesian—it does not allow true probability for X and it violates the Likelihood Principle. Even among frequentist statisticians, it makes some trouble, as the sole source of randomness is induced by the sampling plan. This, in turn, generates likelihood functions which are flat on their support.

(b) Frequentist superpopulation models: Essentially, there is now a probability distribution $F_X(\cdot|\theta)$ on $\mathcal{X}^N$ and inference concentrates on $\theta \in \Theta$. This is again non-Bayesian, as opposed to

(c) Bayesian superpopulation models: There is a formal Bayesian update of densities for θ, allowing computation of predictive densities

$$f_T(t|y) = \int_\Theta f(\theta|y) f_T(t|y,\theta)\, d\theta.$$

This note proposes a return to basic subjectivistic reasoning by considering strict predictivistic, de facto Bayesian models. Under the predictivistic approach, the inference problem is reduced to eliciting the prior $f_T(t)$ for T and updating it by Bayes's formula, yielding

$$f_T(t|y) \propto f_t(t) f(y|T=t).$$

This could also be implemented by assessing first $f_X(x_1, x_2, \ldots, x_N)$, as such marginal distribution decomposes on

$$f_X(x_1, x_2, \ldots, x_N) = \int_{T(X)} f_X(x|T=t) f_T(t)\, dt.$$

It should be noticed that under the predictivistic point of view, there is no distinction between prior and likelihood—both are equally natured components of $f_X(x)$ and are implied by the judgments about quantities to be actually observed.

The predictivistic approach for finite populations has been considered by many authors. The use of operational parameters in Engineering and Physics problems is discussed by Mendel and Kempthorne (1996); Barlow and Mendel (1992) build appropriate models for aging beginning from the judgment of exchangeability for units with respect to lifetime (see also Mendel (1994)); Irony and Pereira (1994) examine the justification for the use of discrete distributions in the area of quality assurance; Bolfarine, Gasco and Iglesias (2003) focus on the prediction of finite population regression coefficients considering certain invariance distributional assumptions. A detailed review on finite population models can be found in Bolfarine and Zacks (1992).

Bayesian inference for operational parameters will be considered based on uniformity and/or exchangeability judgments on sequences of random variables. Interest centers on both the population total and maximum. Some of the distributional results presented have been obtained by other authors. This paper also shows that there are contexts in which the superpopulation approach cannot be employed in inferential problems in a finite population. That is, there are circumstances in which the prior for the operational parameters cannot be obtained from any superpopulation model. In some cases the extension to an infinite population eases the inferential problem. Conditions for extendibility to infinite populations are established for some models. (For further discussion on extendibility, see Bernardo and Smith (1994), Diaconis, Eaton and Lauritzen (1992) and many others.)

This paper is organized as follows. Sections 2–4 deal with the estimation of population totals for discrete and continuous cases and establish conditions to extendibility in almost all situations. Section 5 is devoted to the estimation of population maxima and provides conditions for the extendibility to infinite populations in this case.

2 Estimation of the population total (0–1 case)

Consider the situation where $\mathcal{X} = \{0, 1\}$ and $T = T(X) = \sum_{i=1}^{N} X_i$. Suppose that a sample y is available. Let the prior distribution for T be represented by

$$\text{Prob}\{T = t\} = a_t, \qquad \text{for } t = 0, 1, 2, \dots, N. \tag{2.1}$$

The specification of the distribution of X, given $T = t$, yields the posterior distribution for T, given y, as

$$\begin{aligned} \mathrm{P}\{T = t|y\} &\propto a_t \mathrm{P}\{y|T = t\} \\ &= a_t \sum_{z:S(z)=(x_1,\dots,x_n)} \mathrm{P}\{X = z|T = t\}. \end{aligned} \tag{2.2}$$

It should be emphasized that the sample y is "available", that is, it is not necessarily obtained by a lottery—or, if it is, its sampling plan is irrelevant. That is, sampling plans here play no role. All the randomness present is due to uncertainty about the x_i-values. The use of Bayes's formula is the coherent way of updating static probabilities derived from the original prior distribution $f_X(x_1, \ldots, x_N)$. (See Loschi and Wechsler (2002) for discussion on coherence in the *temporal* context.) It should be noted that the specification of this prior is given by a_t and $\mathrm{P}_X\{x_1, \ldots, x_N | T = t\}$.

As the judgment of exchangeability leads to a conditional uniform distribution on the appropriate set for X, given $T = t$, if $a_t = (N+1)^{-1}$ the following posterior is obtained:

$$\mathrm{P}\{T = t | y\} \propto \frac{\binom{N-n}{t - \sum_{i=1}^n x_i}}{\binom{N}{t}} \mathbf{1}\left\{t \geq \sum_{i=1}^n x_i\right\}. \tag{2.3}$$

This model corresponds to the Bose–Einstein model (Mendel and Kempthorne (1996)). One should notice that the "hypergeometric" term on the posterior in (2.3) results exclusively from the exchangeability assumption. There is no assumption of a "draw without replacement" whatsoever.

It should be noticed, however, that the posterior is exactly the same obtained by assuming $X_1, X_2, \ldots, X_N$ extendible to a sequence of exchangeable random quantities with uniform de Finetti's measure on the interval $(0, 1)$. In fact, if $X_1, X_2, \ldots, X_N$ are exchangeable, the conditional distribution of X, given $T = t$, is uniform. Furthermore, if $X_1, X_2, \ldots, X_N$ are extendible, de Finetti's Representation theorem (de Finetti (1937)) yields a probability measure μ on $[0, 1]$ such that

$$\mathrm{P}\{T = t\} = \int_0^1 \binom{N}{t} \theta^t (1-\theta)^{N-t} \, d\mu(\theta), \tag{2.4}$$

for each $t = 0, 1, \ldots, N$. If μ is uniform on $\Theta = (0, 1)$, $\mathrm{Prob}\{T = t\} = (N+1)^{-1}$.

In conclusion, there is a return to a superpopulation Bernoulli model with uniform de Finetti measure for θ on $(0, 1)$ when the prior a_t is discrete uniform. However, suppose now that $\mathrm{Prob}\{T = 0\} = 0$, and that $\mathrm{Prob}\{T = t\} > 0$, for $t \neq 0$, that is, 0 is removed from the prior support of T. In this case, there is no superpopulation model yielding the posterior distribution for T given in (2.3) as can be observed in the sequel.

Suppose that there is a superpopulation model yielding the same posterior distribution. In other words, assume the existence of a nondegenerate distribution G on $[0, 1]$ such that

$$\mathrm{P}\{T = t\} = \int_0^1 \binom{N}{t} \theta^t (1-\theta)^{N-t} \, dG(\theta),$$

for each $t = 0, 1, \ldots, N$. Then,

$$0 = \mathrm{P}\{T = 0\} = \int_0^1 (1-\theta)^N \, dG(\theta),$$

implying that G is degenerate on 1 and $P(T = N) = 1$.

In conclusion, there are prior distributions $\text{Prob}\{T = t\}$ which do not correspond to any coherent superpopulation model in the sense that the posteriors do not correspond to any one attainable from such a model. There are examples where the support of T is preserved.

In general the problem of establishing which priors do correspond to superpopulation models can be restated in terms of the following question: What prior values a_t satisfy

$$a_t = \int_0^1 \binom{N}{t} \theta^t (1-\theta)^{N-t}\, d\mu(\theta) \qquad \forall t = 0, 1, \ldots, N,$$

for some nondegenerate measure μ on [0, 1]? This is actually Hausdorff's reduced moment problem (Shohat and Tamarkin (1943)) the solution of which characterizes extendible exchangeable sequences (see Iglesias (1993), Feller (1991) and also de Finetti (1937) for the proof of the representation theorem for exchangeable 0–1 variables).

3 Estimation of the population total (discrete case)

Here $\mathcal{X} = \mathcal{Z}_+$ (the nonnegative integers) and $T = T(X) = \sum_{i=1}^N X_i$. Suppose again that a sample y is available and let

$$\text{Prob}\{T = t\} = a_t \qquad \text{for } t \in \mathcal{Z}_+,$$

so that

$$\text{P}\{T = t|y\} \propto a_t \sum_{z:S(z)=(x_1,\ldots,x_n)} \text{P}\{X = z|T = t\}.$$

Suppose also that the distribution of X given $T = t$ is uniform on the set $\{(x_1, \ldots, x_N) \in \mathcal{Z}_+^N : \sum_{i=1}^N x_i = t\}$. This assumption is stronger than mere exchangeability. Standard results from Probability Calculus (Feller (1968), for instance) yield

$$\text{P}\{T = t|y\} \propto a_t \frac{\binom{N-n-t-\sum_{i=1}^n x_i - 1}{t - \sum_{i=1}^n x_i}}{\binom{N-t-1}{t}} \mathbf{1}\left\{t \geq \sum_{i=1}^n x_i\right\}. \tag{3.1}$$

It can be seen from (3.1) that the likelihood is the same when a geometric Bayesian superpopulation model is used. Furthermore, if there is a nondegenerate probability measure μ on [0, 1] such that

$$a_t = \int_0^1 \binom{N-t-1}{t} (1-\theta)^N \theta^t \, d\mu(\theta) \qquad \text{for each } t \in \mathcal{Z}_+, \tag{3.2}$$

then the posterior distribution is exactly the same obtained by assuming $X_1, X_2, \ldots, X_N$, given θ, conditionally independent and geometrically distributed with parameter θ with de Finetti's measure μ for θ.

That is a condition of extendibility for a class of finite exchangeable sequences. The existence of a nondegenerate measure μ satisfying the equations (3.2) can be verified by Hausdorff's theorem (Shohat and Tamarkin (1943)). This can be seen by noticing that the equations in (3.2) have a solution if, and only if, the moment problem

$$c_t = \int_0^t \theta^t \, d\nu(t) \qquad \text{for } t \in \mathcal{Z}_+$$

has a solution for ν, a probability measure on [0, 1], with c_t satisfying the relation

$$\binom{N-t-1}{t}^{-1} a_t = \sum_{r=0}^{N} \binom{N}{r} (-1)^r c_{r+t} = \Delta^N c_t.$$

Hausdorff's theorem shows that the sequence is extendible (in the sense previously defined) if, and only if,

$$\Delta^k a_t = \sum_{r=0}^{k} \binom{k}{r} (-1)^r a^{t+r} \geq 0 \qquad \text{for } k = 1, 2, \ldots, t = 0, 1, \ldots.$$

It is easy to find prior distributions for T which satisfy these conditions. On the other hand, it is not simple to find the de Finetti's measure for which these inequalities are satisfied. Consequently, it is not clear what the advantages are of verifying extendibility, once a prior a_t has been elicited.

The condition of extendibility established is not restricted to judgments of uniform distributions of X, given $T = t$. If, given t, the random variables possess conditional Multinomial$(t; 1/N, \ldots, 1/N)$ distribution, that is, if the sequence $X_1, X_2, \ldots$ is Poissonian (see Wechsler (1993), for instance), a similar condition for a_t can be obtained.

4 Estimation of the population total (continuous case)

Consider $\mathcal{X} = \mathcal{R}_+$ and $T = T(X) = \sum_{i=1}^N X_i$. Let $f(t)$ be the prior density for T and suppose that the distribution of X, given $T = t$, is uniform on the set $\{(x_1, \ldots, x_N) \in \mathcal{R}_+^N : \sum_{i=1}^N x_i = t\}$. This assumption is again stronger than exchangeability. It is known that if $X_1, \ldots, X_N$ are independent random quantities with common exponential (θ) distribution, then the distribution of $X_1, \ldots, X_N$, given $T = t$, is uniform on the appropriate set. Using this fact, the posterior density of T, given y, is obtained as

$$f(t|y) \propto \left(1 - \frac{\sum_{i=1}^n x_i}{t}\right)^{N-n-1} \left(\frac{\sum_{i=1}^n x_i}{t}\right)^{n-1} \frac{1}{t} f(t) \mathbf{1}\left\{t \geq \sum_{i=1}^n x_i\right\}. \tag{4.1}$$

This same approach is considered by Barlow and Mendel (1992) in a lifetime analysis context.

If $f(t) = \frac{am_0^a}{t^{a+1}}\mathbf{1}\{t \geq m_0\}$, that is, if T has a Pareto prior distribution with parameters (a, m_0), the following posterior density is obtained:

$$f\left(t \,\Big|\, \sum_{i=1}^{n} x_i\right) \propto \left(1 - \frac{\sum_{i=1}^{n} x_i}{t}\right)^{N-n-1} \left(\frac{\sum_{i=1}^{n} x_i}{t}\right)^{n+a} \frac{1}{t}\mathbf{1}\{t \geq M\}, \qquad (4.2)$$

where $M = \max\{m_0, \sum_{i=1}^{n} x_i\}$. This posterior does not correspond to any density obtained from an exponential superpopulation model, as the Pareto distribution places mass zero for all $t < m_0$.

For situations in which $m_0 < \sum_{i=1}^{n} x_i$, a probability interval for T, given y, can be constructed by using tables of the Beta distribution with parameters $n + a + 2$ and $N - n$ to obtain quantiles B

$$\text{Prob}\left\{[B_{1-\alpha/2}]^{-1} \sum_{i=1}^{n} x_i < T < [B_{\alpha/2}]^{-1} \sum_{i=1}^{n} x_i \,\Big|\, y\right\} = 1 - \alpha.$$

The inference problems considered in Sections 2–4 refer to the estimation of population totals for finite sequences which, in particular, satisfy the condition of exchangeability. Under a further judgment of linearity of the posterior mean, that is, assuming that

$$E\left(T \,\Big|\, \sum_{i=1}^{n} x_i\right) = a + b\sum_{i=1}^{n} x_i,$$

Ericson's theorem (Ericson (1969)) can be used to determine a and b, which depend on the first and second prior moments of T only. In particular, for the problem introduced in Section 2, if a and b are known, then the prior distribution of T is completely determined. This can be proved by Ericson's theorem and the rule of succession for finite sequences (Zabell (1989); de Finetti (1937)). Diaconis and Ylvisaker (1979) (page 279–280) obtain a similar result for infinite sequences.

Remark. The results presented in this section can be generalized to a more general set in which $\mathcal{X} = \mathcal{R}^d$, d a fixed positive integer. Results from Diaconis and Freedman (1990) yield

$$f(y|t) = \prod_{i=1}^{n} h(x_i) \frac{h^{(N-n)}(t - \sum_{i=1}^{n} x_i)}{h^{(N)}(t)} \mathbf{1}\left\{t \geq \sum_{i=1}^{n} x_i\right\},$$

in which h is a nonnegative, finite and locally Borel-integrable function and $h^{(j)}$ denotes the j-fold convolution of h with itself. Consequently, the posterior distribution of T, given y, is given by

$$f(t|y) \propto \frac{h^{(N-n)}(t - \sum_{i=1}^{n} x_i)}{h^{(N)}(t)} f(t)\mathbf{1}\left\{t \geq \sum_{i=1}^{n} x_i\right\}. \qquad (4.3)$$

If the prior in (4.3) is $f(t) \propto [h^{(2)}(t)]^{-1}\mathbf{1}\{t \geq m_0\}$, for instance, the posterior does not correspond to any density obtained from a superpopulation model defined for the exponential family by Diaconis and Freedman (1990).

For the situation discussed above, $h(x) = \mathbf{1}\{x \geq 0\}$ and the j-fold convolution of h with itself becomes

$$h^{(j)}(s) = \frac{s^{j-1}}{(j-1)!}.$$

5 Estimation of the population maximum (discrete case)

Consider $\mathcal{X} = \mathcal{Z}_+$ and $T = T(X) = X_{(N)}$, the population maximum. A sample y is available and let a_t denote the prior distribution for T with the distribution of X, given $T = t$, uniform on the set $\{(x_1, \ldots, x_N) \in \mathcal{Z}_+^N : \max_{1\leq i\leq N}\{x_i\} = t\}$.

The posterior distribution is given by

$$\operatorname{Prob}\{T = t|y\} = \begin{cases} \dfrac{1}{(t+1)^n}\left\{\dfrac{1}{1+(t/(t+1))^N}\right\}a_t, & \text{if } t = \max\limits_{1\leq i\leq n}\{x_i\}, \\ \dfrac{1}{(t+1)^n}\left\{\dfrac{1-(t/(t+1))^{N-n}}{1-(t/(t+1))^N}\right\}a_t, & \text{if } t > \max\limits_{1\leq i\leq n}\{x_i\}. \end{cases} \tag{5.1}$$

For details in the characterization of the likelihood and connections with de Finetti-style theorems, see Iglesias, Pereira and Tanaka (1998) and Esteves, Wechsler and Iglesias (2004).

Extendibility is obtained if, and only if, there is a nondegenerate probability measure μ on $\mathcal{Z}_+$ such that

$$a_t = \int_{\theta\geq t} \frac{(t+1)^N - t^N}{(t+1)^N}\, d\mu(\theta) \qquad \text{for each } t \in \mathcal{Z}_+. \tag{5.2}$$

It can be noticed from equation (5.2) that the prior for T can be represented as a mixture of discrete uniform distributions. Moreover, the measure μ exists if, and only if, the function defined by

$$h(t) = \left\{\frac{a_t(t+1)^N}{(t+1)^N - t^N} - \frac{a_{t+1}(t+2)^N}{(t+2)^N - (t+1)^N}\right\}, \qquad t \in \mathcal{Z}_+,$$

defines a probability measure on $\mathcal{Z}_+$. Using this fact, it can be shown that if $a_M = 0$ for some $M \in \mathcal{N}$ and $a_t > 0$ for each $t \neq M$, the sequence is not extendible for the uniform superpopulation model. There are examples which preserve the support.

6 Conclusion

This paper shows that Bayesian superpopulation models do not necessarily exist for every prior $f_X(x)$. The operational approach seems to be natural for inference problems involving finite populations since the quantities associated with such problems are potentially observable ones. According to Wechsler (1993), the operational approach for inference removes both the metaphysical character of the parameters and also the asymmetry usually perceived between prior and likelihood as concepts—they are all implied by the prior judgment on observable quantities (Mendel (1994); de Finetti (1937)). The predictivistic approach provides a formal framework for inference also in the situation under which information is provided by *intentional* samples. Strong advancements in this area are given by Kadane (1996), Nagae (2007) and Diniz et al. (2009).

Extendibility to superpopulation models is an interesting approach as long as related inferential problems in finite populations become easy. Otherwise, extendibility is not justified. In this note, conditions for extendibility to infinite populations are established for some models built under uniformity and/or exchangeability judgments. Some of these conditions are not easily verified as they are associated with problems involving moments. A main contribution here is to show that superpopulation models provide solutions to problems involving a finite population only under particular circumstances. Consequently, the operational approach is suggested as the appropriate approach for finite population models.

Acknowledgments

Research support in part by FONDECYT (Chile), Grant 1030588; CNPq (*Conselho Nacional de Desenvolvimento Científico e Tecnológico*) of the Ministry for Science and Technology of Brazil, Grants 304505/2006-4, 472877/2006-2, 472066/2004-8; and FAPESP (*Fundação de Amparo à Pesquisa do Estado de São Paulo*), Grant 2003/10105-2 Pronex: Séries temporais, Análise de Dependência, e Aplicações em Atuária e Finanças.

References

Barlow, R. E. and Mendel, M. (1992). De Finetti-type representations for life distributions. *Journal of the American Statistical Association* **87** 1116–1122. MR1209569

Bernardo, J. and Smith, A. F. M. (1994). *Bayesian Theory*. Wiley, New York. MR1274699

Bolfarine, H. and Zacks, S. (1992). *Prediction Theory for Finite Populations*. Springer, Berlin. MR1214788

Bolfarine, H., Gasco, L. B. and Iglesias, P. (2003). Inference under representable priors for Pearson type II models in finite populations. *Journal of Statistical Planning and Inference* **111** 23–26. MR1955870

Cochran, W. E. (1977). *Sampling Techniques*. Wiley, New York. MR0474575

Daboni, L. and Wedlin, A. (1982). *Statistica-Un'introduzione all'impostazione neo-bayesiana*. Unione Tipografico Editrice Torinese, Torino.

de Finetti, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institut Henry Poincaré (Translated in Kyburg, Jr., H. E.; Smokler, H. E., 1964). Studies in subjective probability* **7** 1–68. MR0179814

de Finetti, B. (1975). *Theory of Probability* **1**. Wiley, New York.

Diaconis, P. and Freedman, D. A. (1990). Cauchy's equation and de Finetti's theorem. *Scandinavian Journal of Statistics* **17** 235–250. MR1092946

Diaconis, P. and Ylvisaker, D. (1979). Conjugate priors for exponential families. *Annals of Statistics* **7** 269–281. MR0520238

Diaconis, P., Eaton, M. L. and Lauritzen, S. L. (1992). Finite de Finetti theorems in linear models and multivariate analysis. *Scandinavian Journal of Statistics* **19** 289–316. MR1211786

Ericson, W. A. (1969). A note on the posterior mean of a population mean. *Journal of the Royal Statistical Society* **31** 332–334.

Esteves, L. G., Wechsler, S. and Iglesias, P. (2004). Some characterizations of uniform models. *Statistics & Probability Letters* **69** 397–404. MR2091759

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications* **1**. Wiley, New York. MR0228020

Feller, W. (1991). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

Fossaluza, V., Diniz, J. B., Pereira, B. B., Miguel, E. and Pereira, C. A. B. (2009). Sequential allocation and balancing prognostic factors in psychiatric clinical trial. *Clinics*. To appear.

Iglesias, P. L. (1993). Finite forms of de Finetti's theorem: A predictivistic approach to statistical inference in finite populations (in Portuguese). Ph.D. thesis, Instituto de Matemática e Estatística, Universidade de São Paulo.

Iglesias, P. L., Pereira, C. A. B. and Tanaka, N. I. (1998). Characterizations of multivariate spherical distributions in l_∞-norm. *Test* **7** 307–324. MR1667002

Irony, T. Z. and Pereira, C. A. B. (1994). Motivation for the use of discrete distributions in quality assurance. *Test* **3** 181–193. MR1365731

Kadane, J. (1996). *Bayesian Methods and Ethics in a Clinical Trial Design*. Wiley, New York.

Loschi, R. H. and Wechsler, S. (2002). Coherence, Bayes's theorem and posterior distributions. *Brazilian Journal of Probability and Statistics* **16** 169–185. MR2008873

Mendel, M. (1994). Operational parameters in Bayesian models. *Test* **3** 195–206.

Mendel, M. and Kempthorne, P. J. (1996). Operational parameters for the use of Bayesian methods in Engineering. *Brazilian Journal of Probability and Statistics* **10** 1–13.

Nagae, C. Y. (2007). Amostragem Intencional (in Portuguese). Master thesis, Instituto de Matemática e Estatística, Universidade de São Paulo.

Shohat, J. A. and Tamarkin, J. D. (1943). *The problems of moments* **1**. Amer. Math. Soc., New York. MR0008438

Wechsler, S. (1993). Exchangeability and predictivism. *Erkenntnis—International Journal of Analytic Philosophy* **38** 343–350. MR1229420

Zabell, S. L. (1989). The rule of succession. *Erkenntnis—International Journal of Analytic Philosophy* **31** 283–321.

P. L. Iglesias
Departamento de Estadística
Pontificia Universidad Católica de Chile
Santiago
Chile
E-mail: pliz@mat.puc.cl

R. H. Loschi
Departamento de Estatística
Universidade Federal de Minas Gerais
30.710-010, Belo Horizonte, MG
Brazil
E-mail: loschi@est.ufmg.br
URL: http://www.est.ufmg.br/~loschi/

C. A. B. Pereira
S. Wechsler
Intituto de Matemática e Estatística
Universidade de São Paulo
05508-090, São Paulo, SP
Brazil
E-mail: cpereira@ime.usp.br
sw@ime.usp.br
URL: http://www.ime.usp.br/~cpereira/
http://www.ime.usp.br/~sw/