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Characterizations of multivariate spherical distributions in l_∞–norm

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Abstract

The objective of this work is to characterize families of distributions which consist of mixtures of the uniform distributions on the surface of the N–sphere in the l_∞–norm. We discuss the characterization through distribution functions and stochastic representations rather than through a measure theoretic approach. Connections with the finite forms of de Finetti-type theorems are considered.

Key Words: Uniform distributions on spheres, finite forms of de Finetti–type theorems, multivariate symmetric distributions.
AMS subject classification: 62A05, 62A15.

1 Introduction

Characterizations of spherical multivariate distributions have been developed in several directions. One of them with interesting connections to the Theory of Robust Inference, is presented in Fang, Kotz and Ng (1990). Another direction is the de Finetti-style theorems related to the foundational aspects of Bayesian Theory (see, for instance, Diaconis and Freedman (1980, 1987), Diaconis, Eaton and Lauritzen (1992), Barlow (1991), Barlow and Mendel (1992), Barlow and Spizzichino (1993)).

In each direction the characterizations of uniform distributions over l_q spheres (and mixtures of them) have been done. Connections to robust Bayesian inference has been considered by Osiewalski and Steel (1993) and modeling in finite populations has been considered by Barlow and Mendel

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(1992). Rachev and Rüschendorf (1991) derived general results characterizing uniform distributions on the surface of the q-sphere in $\mathbb{R}^N$, namely,

$$S_{N,q}(r) = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N : \sum_{i=1}^{N} |x_i|^q = r \right\}$$

and discussed their applications to de Finetti-type theorems. In particular, these last authors showed that those uniform distributions can be characterized through conditional distributions given the sum of $X_1, \ldots, X_N$, i.i.d. positive random variables which satisfy the property that the distribution of $\sum_{i=1}^{n} |X_i|^q / \sum_{i=1}^{N} |X_i|^q$ is a $Beta(n/q, (N-n)/q)$ $(n < N)$. The last condition characterizes the distribution of X_i as a member of an exponential class. The case $q = \infty$, is obtained in a similar way by considering the weak limit of a sequence constructed from functions of quotients of sums. We give characterizations in the last case using a more constructive and geometric approach. Moreover, we exhibit characterizations of uniformity on sets of the form

$$S_N(r_1, r_2) = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N : \min_{1 \le i \le N} \{x_i\} = r_1, \max_{1 \le i \le N} \{x_i\} = r_2 \right\}.$$

In each case we make the connections between the characterizations and the de Finetti-type theorems in both finite and infinite versions.

In Section 2, we discuss the characterizations of uniformity on the surface of N–sphere in the l_∞–norm and the corresponding discrete version. Furthermore, we characterize uniformity on the restricted sphere $S_N(r_1, r_2)$. In Section 3 we relate the characterizations to the finite and infinite forms of de Finetti-style theorems. We omit the details about the finite form results because they are given in a more general measure theoretical framework in Iglesias, Matus, Pereira and Tanaka (1996).

We denote by $\mathcal{B}_N$ the Borel σ–field on $\mathbb{R}^N$ and by $||\cdot||$ the total variation distance, i.e. if P and Q are two probability measures on $(\Omega, \mathcal{A})$ then

$$||P - Q|| = 2 \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.$$

Also, χ^N will denote an N–fold product of a set χ and $\mathbb{Z}_+$, $\mathbb{R}_+$ the non-negative integer and real numbers respectively. By $X_{(n)}$ and $X_{(1)}$ we denote the maximum and the minimum of a sequence $X_1, \ldots, X_n$ respectively.

2 Characterization of Uniformity

In this section we define uniformity geometrically (see Fang, Kotz and Ng (1990)) in the several considered spaces. We get the stochastic representations and from those the characterizations.

2.1 Uniformity on the l_∞-sphere

Let us start with the N–sphere in l_∞–norm defined by

$$S_N(r) = \{(x_1, \dots, x_N) \in \mathbb{R}^N : \max_{1\le i \le N}\{|x_i|\} = r\}, \quad r > 0.$$

The uniform distribution on $S_N(r)$ is defined through the $(N-1)$ dimensional volume as follows. Let

$$\begin{aligned} M_i\left(r(-1)^j\right) &= \{(x_1, \dots, x_N) \in S_N(r) : x_i = r(-1)^j\} \\ &\quad j = 0, 1, \ i = 1, \dots, N, \ r > 0. \end{aligned}$$

Then $S_N(r) = \bigcup_{j=0}^1 \bigcup_{i=1}^N M_i\left(r(-1)^j\right)$. Let $\varphi^i : \mathbb{R}^N \to \mathbb{R}^{(N-1)}$ be defined as $\varphi^i(x_1, \dots, x_N) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$, $i = 1, 2, \dots, N$ and λ be the $(N-1)$-dimensional Lebesgue measure. For $B \in \mathcal{B}_N$ define

$$\mu_i(B) = \lambda(\varphi^i(B \cap M_i(r))) + \lambda(\varphi^i(B \cap M_i(-r)))\ .$$

Definition 2.1. The probability measure $Q_{Nr} : \mathcal{B}_N \to [0,1]$, given by

$$Q_{Nr}(B) = \frac{\sum_{i=1}^N \mu_i(B)}{2N(2r)^{N-1}},$$

is called a uniform probability measure on $S_N(r)$.

To illustrate this definition, let us take $N = 2$, $r = 1$ and $B \subseteq \mathbb{R}^2$. Then $Q_{Nr}(B)$ will be the normalized length of the intersection of B and $S_2(1)$, the border of a square of size 2 and centered at the origin. If we take, for instance, $B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \ge 1\}$ then

$$Q_{21}(B) = 2/8 = 1/4.$$

We notice that the function $\psi : \mathcal{B}_N \times \mathbb{R}_+ \to [0,1]$, defined as $\psi(B, r) = Q_{Nr}(B)$, is a transition function, i.e., fixed $r \in \mathbb{R}_+$, $\psi(., r)$ is a probability measure and fixed $B \in \mathcal{B}_N$, $\psi(B,.)$ is a measurable function in r. The above probability measure is defined on the border of the N-dimensional hypercube centered at the origin. In the next proposition we give a characterization of uniformity through a stochastic representation, similar to Eaton (1981), who defined the uniform distribution on a N-sphere in l_2-norm.

Proposition 2.1. *Let $X_1, X_2, \ldots, X_N$ be independent random variables with common uniform distribution on $(-1,1)$. Let $M_N = \max_{1\le i\le N}\{|X_i|\}$ and $Y_i = r\frac{X_i}{M_N}$, $i = 1,2,\ldots,N$. Then the vector $\boldsymbol{Y} = (Y_1,\ldots,Y_N)$ is uniformly distributed on $S_N(r)$.*

Proof. Let P be the law of $(X_1,\ldots,X_N)$ and Q be the P-law of $\boldsymbol{Y}$. It is clear from the definition of $\boldsymbol{Y}$ that

$$Q(S_n(r)) = P(\boldsymbol{Y} \in S_n(r)) = 1.$$

For $B \in \mathcal{B}_N$ we have

$$Q(B) = P(\boldsymbol{Y} \in B) = \sum_{j=0}^{1}\sum_{i=1}^{N} P(\boldsymbol{Y} \in B \cap M_i(r(-1)^j)).$$

But,

$$P(\boldsymbol{Y} \in B \cap M_i(r)) =$$

$$P\left(\tfrac{X_i}{M_N} = 1,\ r\left(\tfrac{X_1}{M_N},\ldots,\tfrac{X_{i-1}}{M_N},\tfrac{X_{i+1}}{M_N},\ldots,\tfrac{X_N}{M_N}\right) \in \varphi^i(B \cap M_i(r))\right)$$

$$= P\left(X_i > 0,\ r\left(\tfrac{X_1}{|X_i|},\ldots,\tfrac{X_{i-1}}{|X_i|},\tfrac{X_{i+1}}{|X_i|},\ldots,\tfrac{X_N}{|X_i|}\right) \in \varphi^i(B \cap M_i(r))\right),$$

once $(z_1,\ldots,z_{N-1}) \in \varphi^i(B \cap M_i(r))$ implies $|z_j| \le r$ for $j = 1,\ldots,N-1$. Therefore,

$$\begin{aligned} P(\boldsymbol{Y} \in B \cap M_i(r)) &= \int_0^r \lambda(\frac{x}{r}\psi^i(B \cap M_i(r))\frac{1}{2^N}dx \\ &= \tfrac{\lambda(\varphi^i(B\cap M_i(r))}{N\, r^{N-1}2^N}. \end{aligned}$$

Similarly, we can show that

$$P(\boldsymbol{Y} \in B \cap M_i(-r)) = \frac{\lambda(\varphi^i(B \cap M_i(-r)))}{N\, r^{N-1}\, 2^N},$$

concluding the proof. □

Remark 2.1. The uniform distribution on $S_N(r)$ can also be characterized by conditional distribution. In fact, if we allow minor changes in a result presented by Rachev and Rüschendorf (1991) then it can be shown that if $X_1, X_2, \ldots X_N$ are independent random variables with common uniform distribution on $(-1, 1)$, then the conditional distribution of $X_1, \ldots, X_N$ given $\max_{1 \le i \le N}\{|X_i|\} = r$ is uniform on $S_N(r)$ for almost all $r \in [0, 1]$.

The result in the previous remark is also true when $X_1, \ldots, X_N$ is a random vector with absolute continuous density f given by

$$f(x_1, \ldots, x_N) = \psi_N(\max_{1 \le i \le N}\{|x_i|\}),$$

for some ψ_N, a positive function such that f is a density on $\mathbb{R}^N$. The function $\psi_N(\cdot)$ is usually called probability density function generator of the l_∞-spherical distribution. The variable $R = \max_{1 \le i \le N}\{|X_i|\}$ is called radial variable. Moreover if $g(\cdot)$ is the density function of R then its relationship with $\psi_N(\cdot)$ is

$$g(r) = N 2^N r^{N-1} \psi_N(r).$$

From this, a non-negative function $\psi_N(\cdot)$ can be used to define a l_∞-spherical density if and only if

$$\int_0^\infty r^{N-1} \psi_N(r) dr < \infty.$$

In such case, $\psi_N(\cdot)$ satisfies

$$\int_0^\infty r^{N-1} \psi_N(r) dr = \frac{1}{N 2^N}.$$

The quantity in the right-hand side of the last equality corresponds to the (N-1) dimensional volume of $S_N(1)$. See Osiewalski and Steel (1993) for additional discussions about this property in the context of robust Bayesian inference.

Note that $X_1, \ldots, X_N$ with the above assumption are exchangeable. In fact, absolute continuous functions with joint density of the above form are Schur-concave if ψ_N is non-increasing. For non-negative random vector it means that the joint survival distribution is also Schur-concave (Barlow and Spizzichino (1993) and Hayakawa (1993)). This condition is relevant since, as it has recently been shown, it provides a probabilistic model for aging in a subjectivist viewpoint.

In the next proposition we give the distribution of the first n-coordinates of a point uniformly distributed on $S_N(r)$ for $n < N$.

Proposition 2.2. *Let $Y_1, \ldots, Y_N$ be random variables with uniform distribution on $S_N(r)$. Then for $n < N$, the distribution function of $|Y_1|, \ldots, |Y_n|$ is given by*

$$F_r(z_1, \ldots, z_n) = \begin{cases} 0 & \text{if } z_i \leq 0 \text{ for some } i \in \{1, \ldots, n\} \\ \frac{N-n}{N} \prod_{i=1}^{n} \frac{z_i}{r} & \text{if } 0 < z_i < r \text{ for each } i \in \{1, \ldots, n\} \\ \left\{\frac{N-n+k}{N}\right\} \prod_{i=1}^{n} \frac{z_i}{r} & \text{if } z_i = r \text{ for each } i \in I_k \text{ and } 0 < z_i < r \\ & \text{for } i \in J_k,\ k = 1, 2, \ldots, n \\ 1 & \text{if } z_i \geq r \text{ for each } i \in \{1, \ldots, n\}, \end{cases}$$

where $I_k = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots ; n\}$, $J_k = \{1, \ldots, n\} - I_k$.

Proof. The proof follows from Proposition 2.1 and algebraic computations. □

Remark 2.2. It follows from Proposition 2.2 that the distribution function of $Z_n = \max\{|Y_1|, \ldots, |Y_n|\}$ is given by

$$F_r(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \left(\frac{N-n}{N}\right) \left(\frac{z}{r}\right)^n & \text{if } 0 < z < r \\ 1 & \text{if } z \geq r. \end{cases}$$

Rachev and Rüschendorf (1991) proved that if $X_1, \ldots, X_N$ is a sequence of positive independent random variables, then the accumulative distribution function of $X_{n,N} = max_{1 \leq i \leq n}\{X_i\}/max_{1 \leq i \leq N}\{X_i\}$ is $F_1(\cdot)$ for all $n \leq N$ if and only if $X_1 \sim U[0,1]$. Moreover, $X_{n,N}$ is the weak limit as $p \to \infty$ of $Y_{n,N,p}$ where $Y_{n,N,p} = Z^{1/p}$ with $Z \sim Beta(n/p, (N-n)/p)$.

Remark 2.3. A similar characterization can be obtained if we consider the uniform distribution on the surface given by

$$S_N^+(r) = \{(x_1, \ldots, x_N) \in \mathbb{R}_+^N : \max_{1 \leq i \leq N}\{x_i\} = r\}.$$

Let us denote this distribution by $\tilde{Q}_{Nr}$. Note that $\tilde{Q}_{Nr}$ corresponds to the Q_{Nr} law of T, when $T(x_1,\dots,x_N) = (|x_1|,\dots,|x_N|)$. Hence

$$\tilde{Q}_{Nr}(B) = \frac{\sum_{i=1}^{N} \lambda(\varphi^i(B \cap M_i(r)))}{Nr^{N-1}}, \quad B \in \mathcal{B}_N,$$

where $M_i(r) = \{(x_1,\dots,x_N) \in S_N^+(r) : x_i = r\}$.

The characterization by stochastic representation follows as in Proposition 2.1.

Proposition 2.3. *Let $X_1, X_2, \dots, X_N$ be independent random variables with common uniform distribution on (0,1). Let $Y_i = r\frac{X_i}{X_{(N)}}$, $i = 1, 2, \dots, N$. Then the random vector $\mathbf{Y} = (Y_1, Y_2, \dots, Y_N)$ is uniformly distributed on $S_N^+(r)$.*

Proof. It follows from the definition of $\tilde{Q}_{Nr}$ in a similar fashion to Proposition 2.1. The distribution of the n first coordinates of a point uniformly distributed on $S_N^+(r)$ can be obtained directly from the above proposition. □

It follows from Remark 2.3 that the converse to Proposition 2.3 is also true. In words, i.i.d random variables have l_∞-spherical distribution if and only if they are uniformly distributed. In the next section we give a discrete version of that result.

2.2 Discrete Case

Let us consider now the uniform distribution on the N–sphere in l_∞–norm in the discrete case, that is, the uniform distribution on the space

$$S_N^+(r) = \{(x_1,\dots,x_N) \in \mathbb{Z}_+^N : \max_{1\le i\le N}\{x_i\} = r\}, \quad r \in \mathbb{Z}_+.$$

If Q_{Nr} denotes such law, then by simple counting we get

$$Q_{Nr}(x_1,\dots,x_N) = \{(r+1)^N - r^N\}^{-1} I_{\{r\}}(\max_{1\le i\le N}\{x_i\}).$$

Proposition 2.4. *Let $Y_1, \dots, Y_N$ be discrete random variables with uniform distribution on $S_N(r)$. Then for $n < N$, the distribution of $Y_1, \dots, Y_n$ is given by*

$$Q^n_{Nr}(y_1, \dots, y_n) = \begin{cases} \frac{1}{(r+1)^n}\left\{\frac{1-(\frac{r}{r+1})^{N-n}}{1-(\frac{r}{r+1})^N}\right\} & \text{if} \quad \max_{1\le i\le n}\{y_i\} < r \\ \frac{1}{(r+1)^n}\left\{\frac{1}{1-(\frac{r}{r+1})^N}\right\} & \text{if} \quad \max_{1\le i\le n}\{y_i\} = r. \end{cases}$$

Proof. By marginalization we have that

$$Q^n_{Nr}(y_1 \dots, y_n) = \sum_{(z_1, z_2, \dots, z_{N-n}) \in C} Q_{Nr}(y_1, \dots, y_n,\ z_1, \dots, z_{N-n}),$$

with

$$C = \left\{(x_{n+1}, \dots, x_N) \in \mathbb{Z}_+^{N-n} : \max_{1\le i\le N}\{y_1, \dots, y_n, x_{n+1}, \dots x_n\} = r\right\}.$$

The result follows by computing the above summation. □

We now consider the class $\mathcal{C}_N$ consisting of probability measures P obtained by mixing the elements of the family $\{Q_{Nr} : r \in \mathbb{Z}_+\}$ in the radial variable.

Proposition 2.5. *If $P \in \mathcal{C}_N$ then $P^n \in \mathcal{C}_n$ for each $1 \le n < N$, where P^n is a n-dimensional law from P.*

Proof. If $X_1, X_2, \dots, X_N$ are random variables with $P \in \mathcal{C}_N$ then

$$P(X_1 = x_1, \dots, X_n = x_n | X_{(n)} = r_s) = \frac{P(X_1 = x_1, \dots, X_n = x_n) I_{X^{-1}_{(n)}(r_s)}(x_1, \dots, x_n)}{P(X_{(n)} = r_s)},$$

if $P(X_{(n)} = r_s) > 0$ and $r_s \in \mathbb{Z}_+$. Now,

$$\begin{aligned} P(X_{(n)} = r_s) &= \sum_{(x_1, x_2, \dots, x_n) \in X^{-1}_{(n)}(r_s)} P(X_1 = x_1, \dots, X_n = x_n) \\ &= \sum_{(x_1, x_2, \dots, x_n) \in X^{-1}_{(n)}(r_s)} \int Q^n_{Nt}(x_1, \dots, x_n) d\mu_N(t), \end{aligned}$$

where μ_N is the P-law of $X_{(N)}$. But

$$Q^n_{Nt}(z_1, z_2, \ldots, z_n) = Q^n_{Nt}(x_1, \ldots, x_n) \text{ if } \max_{1\leq i\leq n}\{x_i\} = \max_{1\leq i\leq n}\{z_i\}.$$

Therefore,

$$P(X_{(n)} = r_s) = |X^{-1}_{(n)}(r_s)| P(X_1 = x_1, \ldots, X_n = x_n),$$

where $|A|$ denotes the cardinality of A. Consequently,

$$P(X_1 = x_1, \ldots, X_n = x_n | X_{(n)} = r_s) = \frac{1}{|X^{-1}_{(n)}(r_s)|} I_{X^{-1}_{(n)}(r_s)}(x_1, \ldots, x_n).$$

□

Thus, if P^N_θ is the law of n independent random variables uniformly distributed on $\{0, 1, \ldots, \theta\}$, then $P^N_\theta \in \mathcal{C}_N$.

Moreover, the probability measures in $P = \{P^N_\theta : \theta \in \mathbb{Z}\}$ are the unique product probability measures in C_N as can be seen from the next result. That result is a discrete version of the result given by Rachev and Rüschendorf (1991) for the continuous case.

Proposition 2.6. *If $X_1, X_2, \ldots, X_N$ are independent and identically distributed random variables with law $P \in \mathcal{C}_N$ and $N \geq 2$, then $X_1, X_2, \ldots, X_N$ are uniformly distributed.*

Proof. By assumption,

$$P(X_1 = x_1, X_2 = x_2 \mid X_{(2)} = r) = \frac{\prod_{i=1}^2 P(X_1 = x_i) I_{X^{-1}_{(2)}(r)}(x_1, x_2)}{[P(X_1 \leq r)]^2 - [P(X_1 \leq r-1)]^2}$$

if $P(X_{(2)} = r) > 0$. But, $P \in \mathcal{C}_N$ implies that $P_2 \in \mathcal{C}_2$. Hence,

$$\frac{\prod_{i=1}^2 P(X_1 = x_i) I_{X^{-1}_{(2)}(r)}(x_1, x_2)}{[P(X_1 \leq r)]^2 - [P(X_1 \leq r-1)]^2} = \frac{I_{X^{-1}_{(2)}(r)}(x_1, x_2)}{(r+1)^2 - r^2}.$$

Taking $x_1 = x_2 = r$ in the above expression we get

$$\frac{[P(X_1 = r)]^2}{[P(X_1 \leq r)]^2 - [P(X_1 \leq r-1)]^2} = \frac{1}{2r+1}.$$

After some computation this yield

$$\frac{P(X_1 = r)}{P(X_1 \leq r)} = \frac{1}{r+1} \, .$$

Evaluating the above equality at $r = 0, 1, 2, \dots$, we see that

$$P(X_1 = x) = P(X_1 = 0) \quad \text{for each } x \in \mathbb{Z}_+ \, .$$

Adding the fact that P is a probability measure we conclude that there exists a $k \in \mathbb{Z}_+$ so that $P(X_1 > k) = 0$. Therefore,

$$P(X_1 = x) = \frac{1}{k+1} I_{\{0,1,\dots,k\}}(x).$$

□

Remark 2.4. Note that a probability measure P belongs to $\mathcal{C}_N$ if and only if for each $(x_1, \dots, x_N) \in \mathbb{Z}_+^N$,

$$P((x_1, \dots, x_N)) = \varphi_N(\max_{1 \leq i \leq N} \{x_i\}),$$

where φ_N is an appropriate non-negative function.

2.3 Extensions

A natural extension of the uniform distribution on the N-sphere in l_∞-norm is the uniform distribution on the surface of the form

$$S_N(r_1, r_2) = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : \min_{1 \leq i \leq N} \{x_i\} = r_1, \max_{1 \leq i \leq N} \{x_i\} = r_2 \right\},$$

with r_1 and r_2 in $\mathbb{R}$, $r_1 < r_2$ and $N \geq 3$. Set

$$M_{ij}(r_1, r_2) = \{(x_1, \dots, x_N) \in S_N(r_1, r_2) : x_i = r_1, x_j = r_2\}, i \neq j$$

and $i, j \in \{1, 2, \dots, N\}$. Then $S_N(r_1, r_2) = \cup_{i,j \in \{1,\dots,N\}, i \neq j} M_{ij}(r_1, r_2)$. Let $\varphi^{ij} : \mathbb{R}^N \to \mathbb{R}^{N-2}$ be defined by

$$\varphi^{ij}(x_1, \dots, x_N) = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$$

and λ denote the $N-2$ dimensional Lebesgue measure. For B in $\mathcal{B}_N$ define

$$\mu_{ij}(B) = \lambda(\varphi^{ij}(B \cap M_{ij}(r_1, r_2)).$$

Definition 2.2. The probability function $Q_{N(r_1,r_2)} : \mathcal{B}_N \to [0,1]$ defined by

$$Q_{N(r_1,r_2)}(B) = \sum_{i,j \in \{1,2,\dots,N\}} \frac{\mu_{ij}(B)}{N(N-1)(r_2-r_1)^{N-2}}$$

is a uniform probability distribution on $S_N(r_1,r_2)$. The function $Q_{N(r_1,r_2)}$ is a transition function.

In terms of random variables, the uniform distribution can be characterized as follows.

Proposition 2.7. *Let $X_1, X_2 \dots, X_N$ be independent random variables with common uniform distribution on $(0,1)$ and*

$$Y_i = (r_2 - r_1)\left\{\frac{X_i - X_{(1)}}{X_{(N)} - X_{(1)}}\right\} + r_1, \quad \text{with} \quad r_1 < r_2,$$

then $Y = (Y_1, Y_2, \dots, Y_N)$ is uniformly distributed on $S_N(r_1,r_2)$.

Proof. Similar to the proof to Proposition 2.1. □

The distribution of the first coordinates of a point uniformly distributed on the set $S_N(r_1,r_2)$ can be obtained from the previous propositions in a similar fashion as in the other cases. Proposition 2.6 and extensive computations yield the next result.

Proposition 2.8. *Let $Y_1, Y_2 \dots, Y_N$ be random variables with uniform distribution on $S_N(0,1)$ and $Z_{1n} = \min_{1 \le i \le n}\{Y_i\}$, $Z_{2n} = \max_{i \le i \le n}\{Y_i\}$ with $n < N$. Then the joint distribution of (Z_{1n}, Z_{2n}) is given by*

$$F(z_1,z_2) = \begin{cases} 0 & \text{if } z_1 < 0 \text{ or } z_2 < 0 \\ \frac{n(N-n)}{N(N-1)} z_2^{n-1} & \text{if } z_1 = 0 \text{ and } 0 < z_2 < 1 \\ \frac{(N-n)(N-n-1)}{N(N-1)}\{z_2^n - (z_2 - z_1)^n\} & \\ +\frac{n(N-n)}{N(N-1)} z_2^{n-1} & \text{if } 0 \le z_1 \le z_2 < 1 \\ \frac{(N-n)(N-n-1)}{N(N-1)} z_2^n + \frac{n(N-n)}{N(N-1)} z_2^n & \text{if } 0 < z_2 \le z_1 < 1 \\ \frac{(N-n)(N-n-1)}{N(N-1)}\{1 - (1-z_1)^n\} & \\ +\frac{n(N-n)}{N(N-1)}\{1(1-z_1)^{n-1}\} + \frac{n}{N} & \text{if } z_2 = 1, 0 < z_1 < 1 \\ 1 & \text{if } z_1, z_2 \ge 1. \end{cases}$$

Proof. The result follows by using Proposition 2.7 and algebraic computation. □

3 Connections with de Finetti–type theorems

In this section we connect the uniform distributions (and mixtures of them) discussed in the previous section with de Finetti–type theorems. Infinite versions of this type of Theorem characterize the law of infinite sequences in a class of exchangeable random variables as a mixture of conventional parametric models. The purpose is to provide a predictivistic justification (by judgment about observables) for models typically used in infinite populations. However, finite sequences in a class of exchangeable random variables cannot necessarily be represented as a mixture of i.i.d processes. When such representation does not exist finite forms of de Finetti–type theorem have been established. The idea is to estimate the total variation distance between the law of the finite sequence and the mixture of an appropriate product measure law. The statistical interest in this type of result comes from modeling in finite populations. For instance, Barlow and Mendel (1992) use the uniform models on l_q-spheres to provide justification of their analysis of life data in finite populations. Finite form provides an alternative and more constructive form for obtaining the infinite version and from this the relationship with models typically used in infinite populations can be established.

What we are going to do now is to see how the results we have obtained can be used to show some finite forms for the uniform distribution. Essentially, if interest is on proving finite forms in this context, we need to show that the distribution of the n–first coordinates of a point uniformly distributed on $S_N(r)$ (or $(S_N(r_1, r_2))$ is close to the law of n independent random variables with appropriate common uniform distribution. This last distribution is uniform on the interval $(-\theta, \theta)$ in the continuous case, uniform on the set $\{0, 1, \ldots, \theta\}$ in the discrete case and uniform on the interval (θ_1, θ_2) when we consider the surface $S_N(r_1, r_2)$. Let us denote by Q^n_{Nr} or $(Q^n_{N(r_1,r_2)})$ the distribution of the n–first coordinates just mentioned.

Let P^n_θ be the law of n independent random variables uniformly distributed over $(-\theta, \theta)$. Let Q_1 be the Q^n_{N1} - law of $Y_{(n)} = \max_{1\le i\le n}\{|Y_i|\}$, where $(Y_1, \ldots, Y_n)$ has the joint law given by Q^n_{N1} and P_1 be the P^n_1-law of $Y_{(n)}$ when $(Y_1, \ldots, Y_n)$ has the joint law P^n_1.

The total variation distance between Q^n_{Nr} and P^n_r may be computed by adapting the arguments given in Diaconis and Freedman (1987). Observe that $Y_{(n)}$ is a sufficient statistic for $\{P^n_r, r > 0\}$. Furthermore, by computing conditional distributions, it can be shown that $Y_{(n)}$ is also a sufficient statistic for $\{Q^n_{Nr}, r > 0\}$. Thus, by using properties of the total variation distance, we have that

$$||Q^n_{Nr} - P^n_r|| = ||Q^n_{N1} - P^n_1|| = ||Q_1 - P_1||.$$

But,

$$Q_1(w) = \begin{cases} 0 & \text{if } w < 0 \\ (\frac{N-n}{N})w^n & \text{if } 0 \le w < 1 \\ 1 & \text{if } w \ge 1, \end{cases}$$

and

$$P_1(w) = \begin{cases} 0 & \text{if } w < 0 \\ w^n & \text{if } 0 \le w < 1 \\ 1 & \text{if } w \ge 1. \end{cases}$$

Putting these facts together one can show that

$$||Q^n_{Nr} - P^n_r|| = \frac{2n}{N}.$$

We can then get the finite form of de Finetti-type theorem in the continuous case. Let $P_{\mu_n} = \int_{\mathbb{R}+} P^n_\theta d\mu(\theta)$, and C_N be the class of probability measures P on $\mathbb{R}^N$ so that $P = \int_{\mathbb{R}+} Q_{Nr} d\mu(r)$ for some probability measure μ on $\mathbb{R}_+$ and where P_n is a n–dimensional law from $P \in C_N$.

Proposition 3.1. *If P_n and $P_{\mu n}$ are the previously defined probability measures then there exists a probability measure μ on $\mathbb{Z}_+$ such that for each $1 \le n \le N$,*

$$||P_n - P_{\mu n}|| \le \frac{2n}{N}.$$

Proof. It suffices to choose μ as the P–law of $M_N = \max_{1 \le i \le N}\{X_i\}$, where $X_1, \dots, X_N$ has joint law–P in C_N. The result follows from the fact that

$$\left| \int Q^n_{Nr}(A) d\mu(r) - \int P^n_r(A) d\mu(r) \right| \le \int |Q^n_{Nr}(A) - P^n_r(A)| d\mu(r) \le \frac{2n}{N},$$

for any A, Borel subset of $\mathbb{R}^n$. □

Remark 3.1. Using Proposition 2.3 in an analogous way we get the finite form for the discrete case.

Next, we obtain a similar characterization for the distribution of the n–first coordinates of a point uniformly distributed on $S_N(r_1, r_2)$. Let $P^n_{(\theta_1,\theta_2)}$ be the law of n i.i.d. random variables uniform on the interval (θ_1, θ_2) and let $Q^n_{N(r_1,r_2)}$ be the law of the n–first coordinates of a point uniformly distributed on $S_N(r_1 r_2)$. We denote by $\tilde{Q}_1$ the $\tilde{Q}^n_{N(0,1)}$–law of $Z = (Y_{(1)}, Y_{(n)})$, where $Y_1, \ldots, Y_n$ has joint law $Q^n_{N(0,1)}$ and $\tilde{P}_1$ is the corresponding $P^n_{(0,1)}$ –law when $Y_1, \ldots, Y_n$ has joint law $P^n_{(0,1)}$.

Using the fact that Z is a sufficient statistic for both families, $\{P^n_{(\theta_1,\theta_2)} : \theta_1 < \theta_2,\ (\theta_1, \theta_2) \in \mathbb{R}^2\}$ and $\{Q_{N(r_1,r_2)} : r_1 < r_2,\ (r_1, r_2) \in \mathbb{R}^2\}$ we have that

$$||Q^n_{N(r_1,r_2)} - P^n_{N(r_1,r_2)}|| = ||Q^n_{N(0,1)} - P^n_{(0,1)}|| = ||\tilde{Q}_1 - \tilde{P}_1||.$$

The $P_{(0,1)}$–law of Z is given by the density

$$g(z_1, z_2) = \begin{cases} n(n-1)(z_2 - z_1)^{n-2} & \text{if } 0 < z_1 < z_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.8 gives the distribution function associated to $\tilde{Q}_1$. From this we can show that

$$||\tilde{Q}_1 - \tilde{P}_1|| \le \frac{2n(4N - n - 3)}{N(N-1)} \text{ if } 2 \le n \le N.$$

The finite form of a de Finetti style theorem is a consequence of this fact.

Let C_N be the class of probability measures P on $\mathbb{R}^N$ such that

$$P = \int Q_{N(r_1,r_2)} d\mu(r_1, r_2)$$

for some probability measure μ concentrated on $S = \{(x, y) \in \mathbb{R}^2 : x \le y\}$ and let P^n denote the n–dimensional distribution $(n < N)$ from $P \in C_N$. Let $P_{\mu n} = \int P^n_{(\theta_1,\theta_2)} d\mu(\theta_1, \theta_2)$ for some probability measure μ in S. From the last characterization we can show that for $2 \le n \le N$,

$$||P^n - P_{\mu_n}|| \le \frac{2n(4N - n - 3)}{N(N-1)}.$$

Another application of this characterization is the infinite form of de Finetti-type theorems. For the sake of simplicity we only describe the infinite version in the continuous case. The other ones follow in a similar manner.

Now, $\mathcal{C}_N$ is the class of probability measures P on $\mathbb{R}^N$ such that $P = \int_{\mathbb{R}} Q_{Nr} d\mu(r)$ for some probability measure μ on $\mathbb{R}_+$ and P_n is a n-dimensional $(n < N)$ distribution from $P \in \mathcal{C}_N$, so that

$$P_n = \int Q^n_{Nr} d\mu_N(r),$$

where μ_N is the law of $M_N = \max_{1 \le i \le N}\{|Y_i|\}$ with $(Y_1, \ldots, Y_N)$ having the distribution given by P. Note that $P \in \mathcal{C}_N$ implies that the conditional distribution of $(Y_1, \ldots, Y_N)$ given $M_N = r$ is uniform on $S_N(r)$.

Proposition 3.2. *Let $Y_1, Y_2, \ldots$ be an infinite sequence of random variables taking values on $\mathbb{R}$ and let P_n be the law of $Y_1, \ldots, Y_n$. If for each $n \in \mathbb{N}$, $P_n \in \mathcal{C}_n$ then there exists a probability measure μ on $\mathbb{R}_+$ so that for each $(y_1, \ldots, y_n) \in \mathbb{R}^n$,*

$$P(Y_1 \le y_1, \ldots, Y_n \le y_n) = \int_{\mathbb{R}_+} \Pi_{i=1}^n \left\{ \frac{y_i + \theta}{2\theta} I_{(-\theta,\theta)}(y_i) + I_{[\theta,+\infty)}(y_i) \right\} d\mu(\theta).$$

Proof. It is sufficient to show that the sequence $\{\mu_N\}$ is tight and use Proposition 3.1. Additionally, the assumptions imply that

$$P(|Y_1| > k) = \int_{(k,+\infty)} Q^1_{Nr}((r, +\infty)) d\mu_N(r).$$

By using the Remark 2.2 we have that Q^1_{Nr} has distribution function

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{N-1}{N}(\frac{x}{r}) & \text{if } 0 < x < r \\ 1 & x \ge r. \end{cases}$$

Hence,

$$\begin{aligned} P(|Y_1| > k) &= \int_{(k,+\infty)} \left(1 - \left(\frac{N-1}{N}\right) \frac{k}{r}\right) d\mu_N(r) \\ &\ge \int_{\mathbb{R}_+} \left(1 - \frac{k}{r}\right) I_{(k,+\infty)}(r) d\mu_N(r). \end{aligned}$$

The function inside the integral goes to 1 as r goes to ∞. Therefore, given $\delta > 0$, there exists $M = M(\delta) > k_0$ such that if $r > M$

$$\int_{(M,+\infty)} (1-\delta) d\mu_N(r) \geq \eta.$$

From this we conclude that $\{\mu_N\}$ is tight.

By choosing μ as the limit of this subsequence, and using Proposition 3.1 we conclude the proof. □

The tightness argument used above is adapted from Diaconis and Freedman (1987). See, also, Diaconis, Eaton and Lauritzen (1992).

Remark 3.2. If in Proposition 3.2, we replace $\mathcal{C}_n$ by the class of probability measures P on $\mathbb{Z}_+^n$ such that

$$P(Y_1 = y_1, \dots, Y_n = y_n) = \psi_n(\max_{1 \leq i \leq n} \{y_i\})$$

for some appropriate non–negative function ψ_n, then we can derive an infinite version of de Finetti's theorem in the discrete case. By using similar arguments to Proposition 3.2 we can show that there exists a probability measure μ on $\mathbb{Z}_+$ such that for each $(y_1, \dots, y_n) \in \mathbb{Z}_+^n, n \in \mathbb{N}$

$$P(Y_1 = y_1, \dots, Y_n = y_n) = \int \frac{1}{(\theta+1)^n} I_{\{0,1,\dots,\theta\}}(\max_{1 \leq i \leq n} \{y_i\}) d\mu(\theta).$$

Similarly, if in the previous proposition we replace Q_{Nr} by the uniform distribution on $S_N{}^+(r)$ (as in Remark 2.3) then we can obtain the corresponding infinite de Finetti result for the uniform$(0, \theta)$ case, after minor changes. This result has previously been established in Diaconis and Freedman (1984) and in Ressel (1985) using different approaches. See also Rachev and Rüschendorf (1991) for finite and infinite results using a slightly different definition of uniformity. Finally, if $\mathcal{C}_N$ denotes the class of probability measures P on $\mathbb{R}^n$ such that

$$P = \int Q_{N(r_1,r_2)}\, d\mu(r_1, r_2)$$

for some probability measure μ concentrated on $S = \{(x,y) \in \mathbb{R}^2 : x \leq y\}$, then we can derive an infinite version. In fact we can show after some calculations, that if $Y_1, Y_2, \dots$ is a sequence of infinite exchangeable random variables taking values on $\mathbb{R}$ such that $(Y_1, \ldots, Y_n)$ has law $P_n \in C_n$ for each $n \in \mathbb{N}$, then there exists a probability measure μ on S such that for each $n \in \mathbb{N}, B \in \mathcal{B}_n$

$$P((X_1, \ldots, X_n) \in B) = \int_S P^n_{\theta_1,\theta_2}(B) d\mu(\theta_1, \theta_2),$$

where $P^n_{\theta_1,\theta_2}$ is the law of n i.i.d. random variables uniformly distributed on (θ_1, θ_2).

4 Conclusions

We presented characterizations of a family which consists of the radial mixture of the uniform distribution on the surface of the l_∞–norm N–sphere by means of a stochastic characterization of this uniform distribution. We also considered –through the infinite version of de Finetti-type theorems– an important subset of this family which is constructed as a mixture of random vectors with i.i.d. uniform components. In a more applied context, Barlow and Tsai (1995) considered these models in lifetime data analysis.

Extensions of these results in a measure-oriented framework has been studied in a related paper by Iglesias, Matus, Pereira and Tanaka (1996).

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