Reliability Nonparametric Bayesian Estimation in Parallel Systems

Adriano Polpo and Carlos A. B. Pereira

***Abstract*—Relevant results for (sub-)distribution functions related to parallel systems are discussed. The reverse hazard rate is defined using the product integral. Consequently, the restriction of absolute continuity for the involved distributions can be relaxed. The only restriction is that the sets of discontinuity points of the parallel distributions have to be disjointed. Nonparametric Bayesian estimators of all survival (sub-)distribution functions are derived. Dual to the series systems that use minimum life times as observations, the parallel systems record the maximum life times. Dirichlet multivariate processes forming a class of prior distributions are considered for the nonparametric Bayesian estimation of the component distribution functions, and the system reliability. For illustration, two striking numerical examples are presented.**

***Index Terms*—Dirichlet multivariate processes, distribution function, parallel systems, reversed hazard rate, sub-distribution function.**

ACRONYM[1]

HR	hazard rate
RHR	reversed hazard rate
CHR	cumulative hazard rate
CRHR	cumulative reversed hazard rate

NOTATION

$\mathrm{Beta}(a, b)$	distribution Beta with parameters a, and b
$\mathcal{D}(\alpha)$	Dirichlet process with parameter α
$\mathcal{D}_k(\alpha_1, \ldots, \alpha_k)$	Dirichlet multivariate (k-variate) process with parameters $\alpha_1, \ldots, \alpha_k$
$DM_k(\alpha_1, \ldots, \alpha_k)$	Dirichlet multivariate distribution with parameters $\alpha_1, \ldots, \alpha_k$
δ	the last component to fail (that is, if $\delta = j$, then the system has failed because of the j-th component, X_j)
Δ	set of components whose life time is studied (a subset of the non-empty index set $\{1, \ldots, k\}$)
$\mathrm{E}(T\|D)$	conditional expectation of T given D
F_Δ	life distribution function of the set Δ
$F^*_\Delta(t)$	$= \Pr(T \leq t, \delta \in \Delta)$, the sub-distribution function of the risk set Δ
$F(\cdot)$	distribution function of the system
$F_{1,\ldots,k}(\cdot, \ldots, \cdot)$	joint distribution function of all components $(X_1, \ldots, X_k)$
$F_j(t)$	$= \Pr(X_j \leq t)$, the distribution function of the j-th component
$F^*_j(t)$	$= \Pr(T \leq t, \delta = j)$, the sub-distribution function of the j-th component
$F^*_{jn}(t)$	$= (1/n)\sum_{i=1}^n \mathbb{I}(T_i \leq t, \delta = j)$, $j = 1, \ldots, k$, the empirical sub-distribution function of the j-th component
$F_n(t)$	$= \sum_{j=1}^k F^*_{jn}(t)$, the empirical distribution function of the system
$\mathbb{I}(\cdot)$	unit function: $\mathbb{I}(TRUE) = 1$, $\mathbb{I}(FALSE) = 0$
π	product-integral
$\oint g(s)\mathrm{d}s$	integration over disjoint open intervals that do not include the jump points of $g(\cdot)$
k	total number of system components
$\lambda(\cdot)$	hazard rate
$\Lambda(\cdot)$	cumulative hazard rate
$\mu(\cdot)$	reversed hazard rate
$\mu_j(\cdot)$	reversed hazard rate related to a distribution function of the j-th component
$\mathcal{M}(\cdot)$	cumulative reversed hazard rate
$\min(a, b)$	minimum between a and b
$\max(a, b)$	maximum between a and b
n	number of systems in the sample
$\Pr(E)$	probability of event E
ρ_Δ	reliability of the group of components Δ
T	$= \max(X_1, \ldots, X_k)$, the system failure or survival time
$(\mathbf{T}, \boldsymbol{\delta})$	$= \{(T_i, \delta_i) : i = 1, \ldots, n\}$, random sample to be observed
$T^\bullet_{(i)}$	i-th distinct order statistics
X_j	j-th component failure time

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A. Polpo is with the Department of Statistics, Federal University of São Carlos, São Carlos, Brazil (e-mail: polpo@ufscar.br).

C. A. B. Pereira is with the Department of Statistics, University of São Paulo, São Paulo, Brazil (e-mail: cpereira@ime.usp.br).

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[1]The singular and plural of an acronym are always spelled the same.

I. Introduction

In statistical analysis domains, such as engineering, and medicine, censored data are frequently analyzed as part of the statistical inference. Methods that address this kind of data are important in statistical literature; see for instance Cox [1], Breslow & Crowley [2], and Kaplan & Meier [3]. Data representing the life time of engineering systems and their components, or of patients submitted to a particular medical treatment, are common in the statistical literature. Aalen [4], Tsiatis [5], Peterson [6], and Salinas-Torres *et al.* [7] are devoted to nonparametric models for the problem of competing risks, also called series systems.

For parallel systems, there are just a few literature references. Barlow & Proschan [8] precisely define the structure of these systems, and their components. However they only consider cases where the components' distributions and their relations are known. They also describe the concept of coherent structure, and the representation for coherent systems. As a special case, Proschan & Boland [9] study the reliability of a system where the system works if at least j out of k $(j < k)$ components work. The particular case where $j = 1$, a parallel system, is the main objective of the present article. Recall that the series system is the case when $j = k$.

For simplicity, consider two components whose life times, and distributions are denoted by X_1 & X_2, and F_1 & F_2 respectively. One may be interested in the life time, and distribution of the resulting two-component (one of each) system, T, and F, respectively. Considering the two cases above, $T = \min(X_1, X_2)$ for the series system, and $T = \max(X_1, X_2)$ for the parallel system.

The δ random quantity is crucial to obtain the important distributions used in statistical analysis of the parallel system. Considering independence between life times of the components in both parallel and series cases, the distribution of the system can be written as a function of the components' sub-distributions.

The hazard rate (HR) plays an important role in the case of series system analysis. For the parallel case, the corresponding function is the reversed hazard rate (RHR). The concept of RHR appears for the first time in Keilson & Sunita [10]. Important properties of this function can be found in Chandra & Roy [11]. However they were restricted to the case of absolute continuous distributions. Using the product integral, we have generalized the concept of most random variables. See also Block & Savits & Singh [12], and Li & Zuo [13] for more about RHR.

The interrelations among survival, sub-survival, and HR functions are studied in the series system and competing risk literature. The general definition of RHR is introduced in Section II. Section III presents the corresponding interrelations among the parallel system functions: distribution, sub-distribution, and RHR.

To illustrate the main objective of this paper, let us consider the simple case of two components. Suppose we observe a sample of size n of the system reliability; that is, $(\mathbf{T}, \boldsymbol{\delta})$. The statistician's main interest is the estimation of all distribution functions $F_1(\cdot)$, $F_2(\cdot)$, and $F(\cdot)$. Clearly, the estimation of F is a well-known problem in statistics. However, the estimations of F_1, and F_2 requires special tools for censored data. In fact, the censor here is from the right, opposite to the series system whose censor is from the left. This paper treats the general case $k \geq 2$ of a parallel system.

In Section IV, we use Dirichlet multivariate processes, introduced by Salinas-Torres *et al.* [14], to derive Bayesian estimators of all (sub-)distribution functions of the k-component system. To obtain the nonparametric Bayesian estimators, we use the product-integration approach (Gill & Johansen [15]). A survey on Bayesian estimation of survival functions using Dirichlet processes can be found in Ferguson *et al.* [16], and for nonparametric Bayesian analysis in Muller & Quintana [17].

Two numerical illustrative examples are discussed in Section V. The objective is to show how the method properly solves realistic, difficult problems. Section VI describes some future possible research on the subject.

II. Reversed Hazard Rate

The RHR, which is the dual of the HR, is described below. The RHR relates to the distribution function as the HR relates to the survival function. The importance of the RHR to parallel systems is equivalent to that of the HR to the series systems. Denoting probability (density), and distribution functions by f, and F respectively, we write

$$\lambda(t) = \frac{f(t)}{1 - F(t-)},$$

and

$$\mu(t) = \frac{f(t)}{F(t)}.$$

i. Under absolute continuity,

$$F(t) = 1 - \exp\left\{\int_0^t \lambda(y)\mathrm{d}y\right\} = \exp\left\{-\int_t^\infty \mu(y)\mathrm{d}y\right\}. \quad (1)$$

ii. Under discreteness,

$$F(t) = 1 - \prod_{y \leq t}[1 - \lambda(y)] = \prod_{y > t}[1 - \mu(y)]. \quad (2)$$

iii. Under mixing distributions,

$$F(t) = 1 - \exp\{-\Lambda(t)\} = \exp\{-\mathcal{M}(t)\}. \quad (3)$$

Here, $\Lambda(\cdot)$ is the cumulative hazard rate (CHR) and $\mathcal{M}(\cdot)$ is the cumulative reversed hazard rate (CRHR). They are defined as

$$\Lambda(t) = \not\!\!\int_0^t \lambda(y)\mathrm{d}y + \sum_{y \leq t}\{-\ln[1 - \lambda(y)]\}, \quad (4)$$

and

$$\mathcal{M}(t) = \not\!\!\int_t^\infty \mu(y)\mathrm{d}y + \sum_{y > t}\{-\ln[1 - \mu(y)]\}, \quad (5)$$

where $\sum$ is the sum over the set of all jump points of $\lambda(\cdot)$ or $\mu(\cdot)$.

Although the above results can be checked using standard results, the general case is obtained using the product integral described by Gill & Johansen [15], and Andersen *et al.* [18]. If π is the product integral, then item (iii) can be re-written as

$$F(t) = 1 - \underset{0}{\overset{t}{\pi}}[1 - \lambda(y)\mathrm{d}(y)] = \underset{t}{\overset{\infty}{\pi}}[1 - \mu(y)\mathrm{d}(y)]. \quad (6)$$

III. Parallel System

We divide this section in two parts: the first is about relations among the distributions, and sub-distributions in the absolute continuous case; and the second is about general properties of these functions relaxing the restriction of absolute continuity. That is, we allow a finite number of jumps.

A. Relations Among the Distributions and Sub-Distributions in the Absolute Continuous Case

Consider a set of $k(>1)$ positive continuous, s-independent random variables, and T its maximum: $T = \max(X_1, \ldots, X_k)$. Let $F_j(t)$ be the distribution function associated with the random variable X_j, for all $j = 1, \ldots, k$. The j-th sub-distribution function evaluated at a time t is the probability that the whole parallel system survives at most to time t, and that the last component to fail is the j-th component. That is, for the j-th component, the distribution function, and the sub-distribution function evaluated at time t are respectively $F_j(t)$, and

$$F_j^*(t) = \Pr\left\{(X_j \le t)\bigcap_{j\neq\ell}(X_\ell < X_j)\right\}. \quad (7)$$

Recall that the joint distribution function of $(X_1, \ldots, X_k)$ evaluated at $t_1, \ldots, t_k$ is

$$F_{1,\ldots,k}(t_1, \ldots, t_k) = \Pr\left\{\bigcap_{j=1}^{k}(X_j \le t_j)\right\}. \quad (8)$$

Here, it is assumed that this function has a continuous partial derivative over all arguments.

The discussion below is used to establish the connection between (7), and (8).

Theorem 1: The derivative of $F_j^*(t)$, $(\mathrm{d}F_j^*(t)/\mathrm{d}t)$, is equal to the partial derivative of $F_{1,\ldots,k}(t_1, \ldots, t_k)$ at the j-th component, $\partial F_{1,\ldots,k}(t_1, \ldots, t_j, \ldots, t_k)/\partial t_j$, evaluated at $t_1 = t_2 = \ldots = t_k = t$.

Theorem 1 indicates a strong relationship between the set of distribution functions, and the set of sub-distribution functions:

$$F_j^*(t) = \int_0^t \left[\left.\frac{\partial F_{1,\ldots,k}(y_1, \ldots, y_k)}{\partial y_j}\right|_{y_1=y_2=\ldots=y_k=y}\right]\mathrm{d}y.$$

Because the life of the components are mutually s-independent,

$$F_{1,\ldots,k}(t_1, \ldots, t_k) = \prod_{\ell=1}^{k} F_\ell(t_\ell), \quad (9)$$

and

$$\frac{\mathrm{d}F_j^*(t)}{\mathrm{d}t} = \mu_j(t)\prod_{\ell=1}^{k} F_\ell(t), \quad (10)$$

where μ_j is the RHR of the j-th component:

$$\mu_j(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln F_j(t). \quad (11)$$

From (11), one can write

$$F_j(t) = \exp\left\{-\int_t^\infty \mu_j(y)\mathrm{d}y\right\}. \quad (12)$$

Letting $\mu(y) = \sum_{\ell=1}^{k}\mu_\ell(y)$, (10) becomes

$$\frac{\mathrm{d}F_j^*(t)}{\mathrm{d}t} = \mu_j(t)\exp\left\{-\int_t^\infty \mu(y)\mathrm{d}y\right\}. \quad (13)$$

Taking now the sum for $j = 1, \ldots, k$ in both sides of (13), we obtain

$$\sum_{j=1}^{k}\frac{\mathrm{d}F_j^*(t)}{\mathrm{d}t} = \mu(t)\exp\left\{-\int_t^\infty \mu(y)\mathrm{d}y\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\exp\left\{-\int_t^\infty \mu(y)\mathrm{d}y\right\}. \quad (14)$$

Consequently,

$$\sum_{j=1}^{k}F_j^*(t) = \exp\left\{-\int_t^\infty \mu(y)\mathrm{d}y\right\}, \quad (15)$$

which combined with (13) leads to

$$\mu_j(t) = \frac{\mathrm{d}F_j^*(t)/\,\mathrm{d}t}{\sum_{\ell=1}^{k}F_\ell^*(t)}. \quad (16)$$

Finally, (12) implies

$$F_j(t) = \exp\left\{-\int_t^\infty \frac{\mathrm{d}F_j^*(y)}{\sum_{\ell=1}^{k}F_\ell^*(y)}\right\}. \quad (17)$$

Consider that a sample of n identically distributed systems is observed. That is, the random sample $\{(T_i, \delta_i) : i = 1, \ldots, n\}$ is observed. With such a sample, all sub-distribution functions can be estimated. However, this is not the case for joint distribution functions. Hence, the crucial question is "How can information about the joint survival distribution of components be extracted from the sub-distribution functions?". The set of equations just derived partially answers this question. In the next section, using (17), we obtain a strong relation between the distribution function, and the sub-distribution functions for any case, not only in the absolute continuous case.

B. General Properties

This section is dedicated to parallel system results, and properties. For simplicity, we first consider the problem of $k = 2$, a system with two components, and later we extend the results for any $k (> 1)$.

Let (X_1, X_2) be a pair of s-independent positive random variables representing the life of two components of a parallel system. Represent by $F_1(\cdot)$, and $F_2(\cdot)$ respectively, their marginal distribution functions. The life time of the system is represented by $T = \max(X_1, X_2)$; and the indicator of the failed component is $\delta = 1$ if $T = X_1$, and $\delta = 2$ if $T = X_2$. The restriction here is that the two sets of jump points of F_1, and of F_2 have no common points.

The following properties can be proved.

Property 1: The sub-distribution functions can be expressed using the marginal distribution functions of both components:

$$F_1^*(t) = \int_0^t [F_2(y)] \cdot [\mathrm{d}F_1(y)]. \tag{18}$$

Property 2:

a. $F_1^*(+\infty) = \Pr(\delta = 1) = \Pr(X_1 > X_2)$;
b. $F_2^*(+\infty) = \Pr(\delta = 2) = \Pr(X_2 > X_1)$;
c. $F_1^*(+\infty) + F_2^*(+\infty) = 1$.

Property 3: The sub-distribution functions F_1^*, and F_2^* determine the distribution function of the system.

$$\begin{aligned} F_1^*(t) + F_2^*(t) &= \Pr\left(\max(X_1, X_2) \le t\right) \\ &= F_1(t)F_2(t) = \Pr(T \le t) = F(t). \end{aligned}$$

Property 4: The set of jump points of F_j^* and F_j are equal, $j = 1, 2$. Because F_1, and F_2 have disjoint sets of jump points, so have F_1^*, and F_2^*.

Property 5: If $\mathcal{C}_j[\mathcal{C}_j^*]$ is the support of F_j $[F_j^*]$, and by $Y_j = \inf \mathcal{C}_j$ $[Y_j^* = \inf \mathcal{C}_j^*]$, $j = 1, 2$, then $\max(Y_1, Y_2) = \min(Y_1^*, Y_2^*) = t^*$. In words, at least one of the two sub-distribution functions F_1^* and F_2^* is positive for $t \ge t^*$, and both are zero for $t < t^*$.

Note that (17) is the "inverse" of (18). Unfortunately, this expression does not work for the case with jump points. To obtain a version of (17) in the presence of jumps, we introduce the following definition and theorem.

Definition 1: The function $\Phi(F_1^*, F_2^*, t)$ based on the sub-distributions $F_1^*(\cdot)$, and $F_2^*(\cdot)$ is

$$\Phi\left(F_1^*, F_2^*, t\right) \equiv \exp\left[\oint_t^\infty \frac{-\mathrm{d}F_1^*(y)}{F_1^*(y) + F_2^*(y)} + \sum_{y>t} \ln\left(\frac{F_1^*(y-) + F_2^*(y-)}{F_1^*(y+) + F_2^*(y+)}\right)\right], \tag{19}$$

where $\sum$ is the sum over the set of all jump points of F_1^*.

The next result, although restricted to $k = 2$, extends expression (17) in the sense that it can include disjoint jump points.

Theorem 2: The sub-distribution functions F_1^*, and F_2^* determine (uniquely) the distribution function F_1 for $t \ge t^*$. That is,

$$F_1(t) = \Phi\left(F_1^*, F_2^*, t\right). \tag{20}$$

Recall that if $F_1(t^*) = 0$, then $F_1(t) = 0$ for all $t < t^*$; the smallest possible failure time of component 1 is t^*.

The above strong result, together with property 1, indicates that a one-to-one correspondence between (F_1, F_2) and (F_1^*, F_2^*) exists.

We have established the relation between sub-distribution, and distribution functions for a parallel system with two components. The generalization for a system with k components in parallel is straightforward as follows.

Exploring only the j-th component, one should just take $\Delta = \{j\}$, and its complement Δ^c. There are analogous properties to those presented before. For instance, we have three properties:

Property 6: $F_\Delta^*(+\infty) = \Pr(T \le +\infty, \delta \in \Delta) = \Pr(\delta \in \Delta)$, $F_\Delta^*(\infty) + F_{\Delta^c}^*(\infty) = 1$, and $\rho_\Delta := F_\Delta^*(+\infty) = \Pr(\delta \in \Delta)$.

Property 7: $F_\Delta^*(t) + F_{\Delta^c}^*(t) = \Pr(T \le t) = F(t)$.

Property 8: The discontinuity points of F_Δ^*, and F_Δ are equal. Because F_Δ, and F_{Δ^c} have disjoint sets of jump points, so have F_Δ^*, and $F_{\Delta^c}^*$.

For $F_\Delta(t) = \Pr(\max_{j\in\Delta}(X_j) \le t)$, Theorem 3 generalizes Theorem 2.

Theorem 3: The sub-distribution functions determine (uniquely) the distribution functions for $t \ge t^*$. That is,

$$F_\Delta(t) = \Phi\left(F_\Delta^*, F_{\Delta^c}^*; t\right), \quad \text{for } t \ge t^*, \tag{21}$$

where Φ is defined by (19). If $F_\Delta(t^*) = 0$, then $F_\Delta(t) = 0$ for all $t < t^*$.

Before concluding this section, we emphasize that (21) is a very strong result of its dual, obtained for the series system in Salinas-Torres *et al.* [7].

IV. Bayesian Analysis

The objective of this section is to describe a Bayesian reliability approach to parallel systems. We derive a Bayesian estimator of the distribution function F_Δ, and define the multivariate Dirichlet process. Recall that we are dealing with the life of k components. Hence, for each component, we define a Dirichlet process for its reliability. Having the Dirichlet processes for all components as their prior processes, and considering the observed data, the posterior multivariate Dirichlet process results. This method also generates the posterior processes for all sub-distribution functions of the components. From these processes, we obtain the Bayesian nonparametric estimation for all sub-distribution functions. Using the results of Section III, we present the Bayesian estimator for the system, and all components' reliabilities.

For Dirichlet (univariate) process properties, see Ferguson [19]. Here, the finite positive measure used in the definition of Dirichlet processes is a probability measure multiplied by a known constant. The multivariate Dirichlet processes, defined in Salinas-Torres *et al.* [14], may have the following simplified version.

Definition 2: Let Ω be a sample space, $\alpha_1, \ldots, \alpha_k$ be finite positive measures defined over Ω, and $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_k)$ be a random vector having a Dirichlet distribution with parameters $(\alpha_1(\Omega), \ldots, \alpha_k(\Omega))$. Consider k Dirichlet processes,

$P_1,\ldots,P_k$, with $P_j \sim \mathcal{D}(\alpha_j)$, $j=1,\ldots,k$. All these processes, and $\boldsymbol{\rho}$ are $(k+1)$ mutually s- independent random quantities. Define $\mathbf{P}^* = (P_1^*,\ldots,P_k^*) = (\rho_1 P_1,\ldots,\rho_k P_k)$. The $\mathbf{P}^*$ is a Dirichlet multivariate (k-variate) process with parameter measures $\alpha_1,\ldots,\alpha_k$; that is, $\mathbf{P}^* \sim \mathcal{D}_k(\alpha_1,\ldots,\alpha_k)$.

In the context of parallel systems, consider $\Omega = (0,\infty)$, $\rho_j = \Pr(\delta = j)$, and $P_j^*(t) = \Pr(T \le t|\delta = j)$, $j=1,\ldots,k$. Then the prior for $\mathbf{F}^* = (\rho_1 P_1^*,\ldots,\rho_k P_k^*)$, the vector of components' sub-distribution functions, is $\mathbf{F}^* \sim \mathcal{D}_k(\alpha_1,\ldots,\alpha_k)$.

The induced prior for F_Δ^* is given by

$$F_\Delta^*(t) \sim \mathrm{Beta}\left(c_k F_{\Delta,0}^*(t); c_k\left(1 - F_{\Delta,0}^*(t)\right)\right), \quad t>0, \quad (22)$$

where $\mathrm{Beta}(a,b)$ is the distribution Beta with parameters a and b, $c_k = \sum_{j=1}^k \alpha_j(0,\infty)$, and $F_{\Delta,0}^*(t) = \sum_{j\in\Delta}\alpha_j(0,t]/c_k$ is the prior mean of F_Δ^*. Also, $F_0(\cdot) = F_{\Delta,0}(\cdot) + F_{\Delta^c,0}(\cdot)$ is the prior mean of $F(\cdot)$.

Note that $F_\Delta^*(\cdot)$ is a Beta process on $[0,\infty)$ with s-independent increments of the type (Hjort [20])

$$\mathrm{d}F_\Delta^*(t) \sim \mathrm{Beta}\left(c_k \mathrm{d}F_{\Delta,0}^*(t), c_k\left(1 - \mathrm{d}F_{\Delta,0}^*(t)\right)\right). \quad (23)$$

The following result gives the prior mean of the distribution function F_Δ in terms of the prior mean of its associated CRHR $\mathcal{M}_\Delta$.

Lemma 1: Suppose that F_Δ and F_{Δ^c} have no common discontinuities. Under the prior (22), for $F_\Delta^*(\cdot)$, the prior mean of the distribution function F_Δ is given by, for each $t>0$,

$$F_{\Delta,0}(t) := \mathrm{E}\left[F_\Delta(t)\right] = \underset{t}{\overset{\infty}{\pi}}\left(1 - \mathrm{d}\mathcal{M}_{\Delta,0}(s)\right),$$

where $\mathcal{M}_{\Delta,0}(s) := \mathrm{E}[\mathcal{M}_\Delta(s)]$ is the prior mean of $\mathcal{M}_\Delta$ associated to the distribution function F_Δ.

Proof: See Salinas-Torres *et al.* [7]. ■

The posterior distribution of $\mathbf{F}^*$ is an updated Dirichlet multivariate process where $\mathbf{F}^*(t)|n\mathbf{F}_n^*(t) \sim DM_k(\alpha_1(0,t]+nF_{1n}^*(t),\ldots,\alpha_k(0,t] + nF_{kn}^*(t))$; see Salinas-Torres *et al.* [14].

Let

$$F_{\Delta,n}^*(t) = \sum_{j\in\Delta} F_{jn}^*(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}(T_i \le t, \delta \in \Delta)$$

be the empirical sub-distribution function associated with the risk subset Δ.

Let $p_n = c_k/(c_k+n)$. The Bayesian estimators of $F_\Delta^*(\cdot)$, and $F(\cdot)$ are given by

$$\widehat{F}_\Delta^*(t) = p_n F_{\Delta,0}^*(t) + (1-p_n)F_{\Delta,n}^*(t), \quad (24)$$

and

$$\widehat{F}(t) = \sum_{j=1}^k \widehat{F}_j^*(t) = \widehat{F}_\Delta^*(t) + \widehat{F}_{\Delta^c}^*(t). \quad (25)$$

These Bayesian estimators are strongly consistent. For instance, using Glivenko Cantelli Theorem (cf. Billingsley [21], page 275), and the fact that p_n decreases to 0, $p_n \downarrow 0$, it can be shown that $\widehat{F}_\Delta^*$ converges to F_Δ^* uniformly with probability 1.

If $\alpha_j(0,\infty) < \infty, \forall j \in \Delta$, the Bayesian estimator of $\rho_\Delta = \Pr(\delta \in \Delta)$ is given by

$$\widehat{\rho}_\Delta = \lim_{t\uparrow\infty} \widehat{F}_\Delta^*(t) = \frac{\sum_{j\in\Delta}\alpha_j(0,\infty)}{n + \sum_{j=1}^k \alpha_j(0,\infty)} + \frac{\sum_{j=1}^n \mathbb{I}(\delta_j \in \Delta)}{n + \sum_{j=1}^k \alpha_j(0,\infty)}. \quad (26)$$

Let the $m\ (\le n)$ distinct order statistics of T be $T_{(1)}^\bullet < \cdots < T_{(m)}^\bullet$. Set $N_i = \sum_{\ell=1}^n \mathbb{I}(T_\ell < T_{(i)}^\bullet)$, and $d_{\Delta i} = \sum_{\ell=1}^n \mathbb{I}(T_\ell = T_{(i)}^\bullet, \delta_\ell \in \Delta)$, $i=1,\ldots,m$. Define

$$I_\Delta(t) = \exp\left\{\frac{-1}{\sum_{j=1}^k \alpha_j(0,\infty) + n}\sum_{j\in\Delta}\int_t^\infty \frac{\mathrm{d}\alpha_j(0,s]}{\widehat{F}(s)}\right\}, \quad (27)$$

and

$$\Pi(t) = \prod_{i:T_{(i)}^\bullet > t} \frac{\sum_{j=1}^k \alpha_j\left(0, T_{(i)}^\bullet\right] + N_i}{\sum_{j=1}^k \alpha_j\left(0, T_{(i)}^\bullet\right] + N_i + d_{\Delta i}}. \quad (28)$$

The main result of this paper is presented next.

Theorem 4: Suppose that $\alpha_1(0,\cdot),\ldots,\alpha_k(0,\cdot)$ are continuous on (t,∞), for each $t>0$, and F_Δ and F_{Δ^c} have no common discontinuities. Then, for $t \ge T_{(1)}$,

$$\widehat{F}_\Delta(t) = \Phi\left(\widehat{F}_\Delta^*, \widehat{F}_{\Delta^c}^*, t\right) = I_\Delta(t)\Pi(t) \quad (29)$$

is the nonparametric estimator of $F_\Delta(t)$ based on posterior mean.

V. Numerical Examples

This section presents two examples related to the estimation of distribution functions involved in a three-component parallel system. The failure times for the observations are measured in hours.

Example 1: We obtained 100 observations from 3 simulated processes: the first component (X_1) is distributed as exponential with mean 1; the second component (X_2) is distributed as lognormal with mean 1 and standard deviation 0.4; and the third component (X_3) is distributed as a composition of discrete, and continuous distributions

$$F_3(x) = \begin{cases} 0.1466 & x \le 0.005 \\ 1 - \exp\left\{-\left(\frac{x}{0.5}\right)^{0.4}\right\} & 0.005 < x \le 5.5 \\ 1 & x > 5.5. \end{cases}$$

Note that, by considering the parallel system, the presented Bayesian estimators are based on 100 observations of (T,δ). The simulated values are listed in Appendix II.

The objective of this example is to illustrate the efficiency of the ideas just developed. We have the following situation: 92 observations of component one, 22 of component two, and 86 of component three are all censored. Recall that we have included discontinuity points, which are unusual. For this example, we have a very challenging case with few registered observations for components one and three.

The estimation steps are as follows.

1 *Defining Priors:* For simplification, we choose the exponential distribution, with mean $\mu = 1$, as the prior measures (recall that α_1, α_2, and α_3 are finite measures). That is, for $j = 1, 2, 3$, $\alpha_j(0, s] = 1 - e^{-s}$, and clearly $\alpha_j(\Omega) = 1$. We note that this prior is not very informative because the measure of the whole parameter space is only one.
This means that the prior does not overweight the posterior mean, the Bayesian estimator. In fact, for the three distributions, we consider *a priori* three i.i.d. Dirichlet processes with their measures defined by an exponential distribution with mean $\mu = 1$.

2 *Obtaining Posteriors:* The posterior processes for the sub-distribution functions are Dirichlet processes measure defined by an exponential distribution with mean $\mu = 1$, plus the empirical sub-distribution function multiplied by the sample size. That is, for the j-th component, the posterior measure of the Dirichlet process is $(1-e^{-t}) + \sum_{i=1}^{100} \mathbb{I}(T_i \leq t, \delta = j)$.

3 *Computing Reliabilities:* Equation (25) provides the estimator of the system distribution function. In this example, we have

$$\widehat{F}(t) = \frac{100 F_{100}^{*}(t) + 3(1 - e^{-t})}{103}.$$

Now, taking $\Delta = j$, $j = 1, 2, 3$, the distribution function estimator of the j-th component is obtained from (29). We estimate ρ_j by $\widehat{\rho}_j$ using (26).
To perform the necessary calculations, some adjustments have to be done. In the expression (29) of the estimator $\widehat{F}_{\Delta}(t)$, the integral $\int_t^{\infty} (\mathrm{d}\alpha_j(0, s]/\widehat{F}(s))$, $j \in \Delta$, needs numerical approximation. We use the classical method of Runge-Kutta for the solution of $(\mathrm{d}y/\mathrm{d}x) = g(x, y)$, $y(x_0) = y_0$.
The relevant expressions are

$$\begin{aligned} y_{m+1} &= y_m + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ k_1 &= g(x_m, y_m), \\ k_2 &= g\left(x_m + \frac{h}{2}, y_m + \frac{hk_1}{2}\right), \\ k_3 &= g\left(x_m + \frac{h}{2}, y_m + \frac{hk_2}{2}\right), \\ k_4 &= g(x_m + h, y_m + hk_3). \end{aligned}$$

Note that y_m is the resulting value for the solution in x_m, and $x_{m+1} - x_m = h$, where $g(x, y) = f(x)$. These formulas reduce to

$$y_{m+1} = y_m + \frac{h}{6}\left[f(x_m) + 4f\left(x_m + \frac{h}{2}\right) + f(x_m + h)\right]. \tag{30}$$

This solution is equivalent to

$$\inf_{x_m}^{x_{m+1}} f(x)\mathrm{d}x \approx \frac{h}{6}\left[f(x_m) + 4f\left(x_m + \frac{h}{2}\right) + f(x_m + h)\right], \tag{31}$$

which is the result of the application of Simpson rule to the interval $[x_m, x_{m+1}]$. For more details and other numerical integration methods, see Davis & Rabinowitz [23].

Fig. 1 presents the estimates of the four distribution functions: components 1, 2, and 3; and the system. In all plots, the *true* distribution functions (dashed lines) are also illustrated.

The conditional reliabilities of the components relatively to the system are $\widehat{\rho}_1 = \Pr(\max(X_1, X_2, X_3) = X_1) \cong 0.0874$, $\widehat{\rho}_2 = \Pr(\max(X_1, X_2, X_3) = X_2) \cong 0.7670$, and $\widehat{\rho}_3 = \Pr(\max(X_1, X_2, X_3) = X_3) \cong 0.1456$.

Surprisingly, despite the problematic amount of censored observations, and discontinuity presence, the graphics show good performance, and useful estimates.

The previous example shows good performances, even for components 1 and 3, which are under inferior conditions compared to component 2.

In the following example, to better understand the problem of discontinuity, we modify Example 1 by increasing the reliability of the third component.

Example 2: Consider Example 1 with the following modifications.

$$F_3(x) = \begin{cases} 0.1175 & x \leq 2 \\ 1 - \exp\left\{-\left(\frac{x}{4}\right)^3\right\} & 2 < x \leq 5 \\ 1 & x > 5. \end{cases}$$

The situation now is as follows. We have 97 observations of component one, 57 of component two, and 46 of component three, all of which are censored. The conditional reliabilities of the components relatively to the system are $\widehat{\rho}_1 = \Pr(\max(X_1, X_2, X_3) = X_1) \cong 0.0388$, $\widehat{\rho}_2 = \Pr(\max(X_1, X_2, X_3) = X_2) \cong 0.4272$, and $\widehat{\rho}_3 = \Pr(\max(X_1, X_2, X_3) = X_3) \cong 0.5340$. For the estimated distribution functions, see Fig. 2.

Again, despite the presence of discontinuity and censoring, we obtain good performance, and useful estimates.

A. Remarks

1) The examples were conveniently chosen to illustrate the application of the Bayesian nonparametric methodology. Clearly, when dealing with parallel systems, if a component reliability is drastically greater than another, the latter will have more uncensored lifetime observations than the former. The estimation of the life distribution of the former might be inappropriate because of a lack of observations. For instance, in Example 2, we generated another 30 observations of the system. As expected, with this new sample of size 30, component 1 only had 2 uncensored observations. The other two components had 8, and 20 uncensored observations. In fact, except for component 3, the other two had few uncensored observations. The inferences for the component distributions in this case are not as good as in the original Example 2, where the numbers of uncensored observations of components 2 and 3 were large enough to improve the inference of component 1 (that had only 3 uncensored observations). On the other hand, if one focuses only on the system distribution, then the inference seems adequate in this small sample of size 30, as shown in Fig. 3.

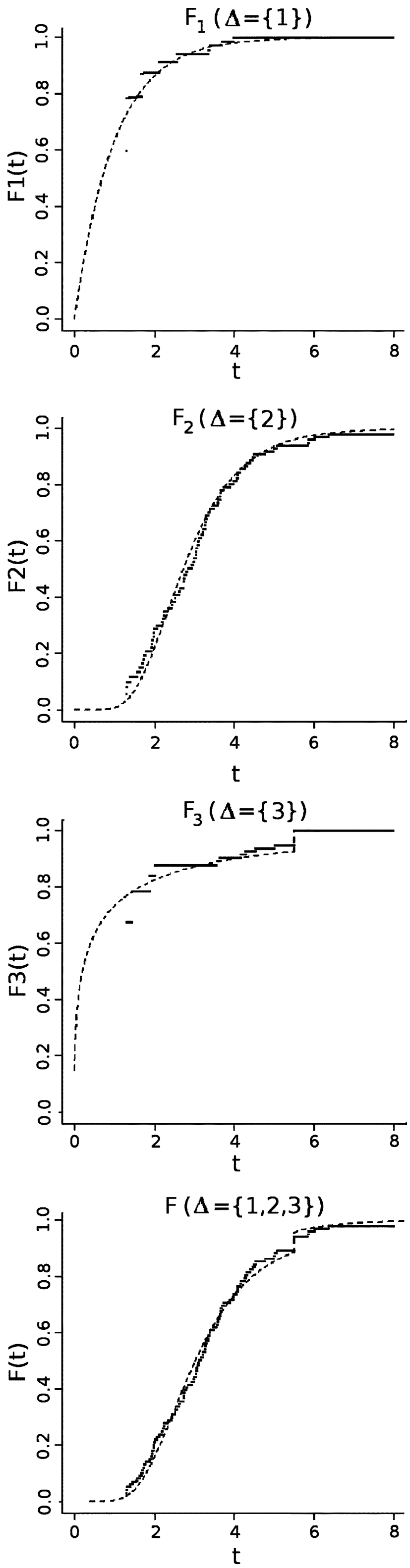

Fig. 1. Estimates for the Example 1.

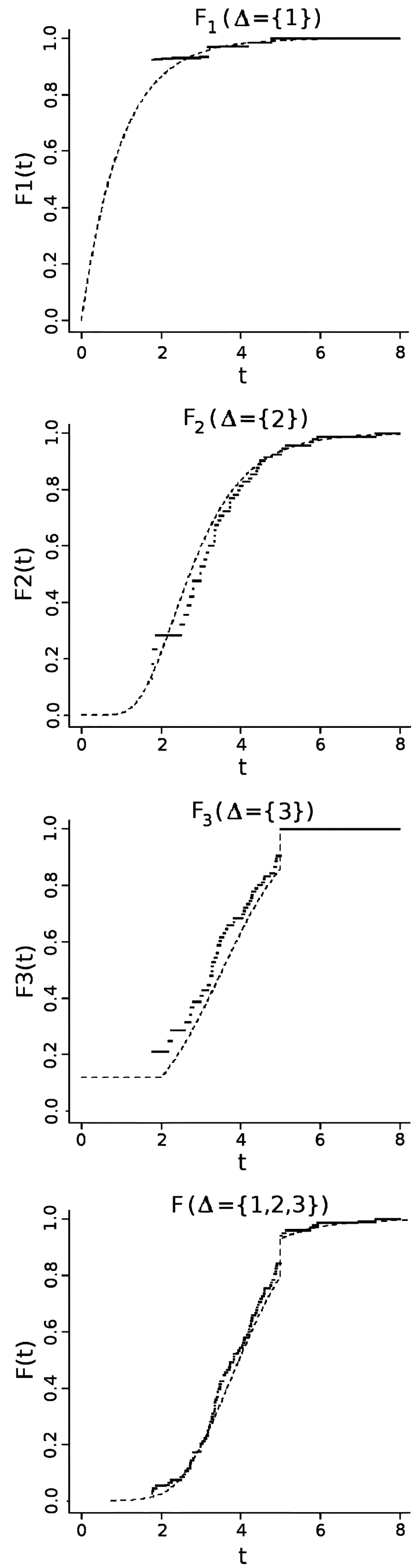

Fig. 2. Estimates for the Example 2.

2) We have chosen a prior that we believe has low influence on the final estimation. In fact, if a specialist has enough information to consider more appropriate priors, he only has to follow the same steps described in this section to obtain more adequate estimators.

3) The judgments we made while analysing the plots are a consequence of the many simulated results performed with alternative distributions. We have chosen two hard examples that would impose difficulties for any possible statistical methodology.

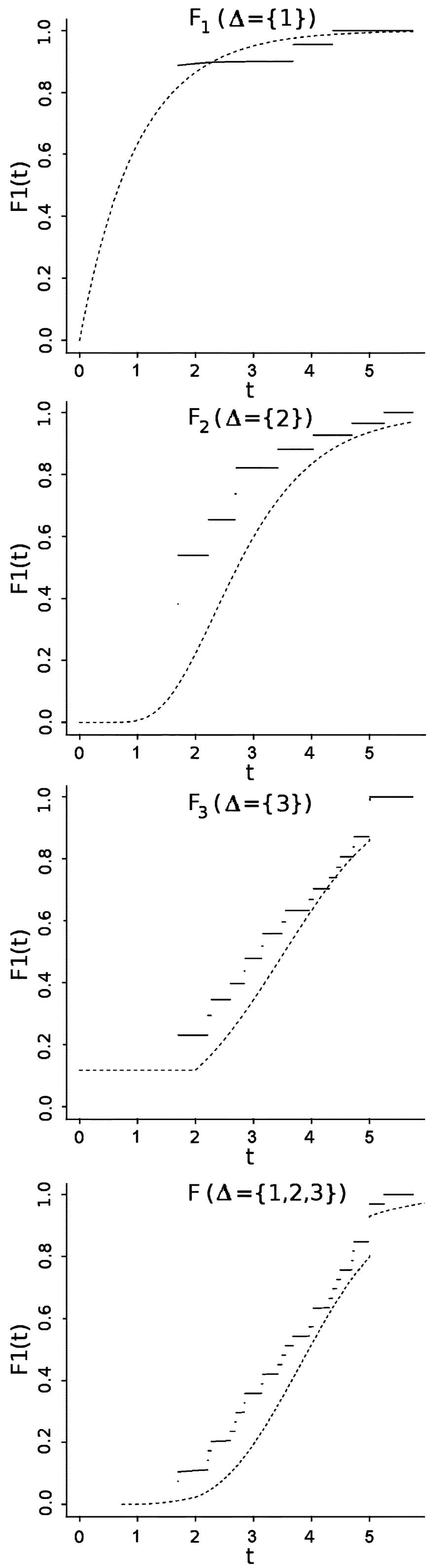

Fig. 3. Example 2 with sample size 30.

4) We consider cases with discontinuity points. The reason is that there are real situations for which the occurrence of discontinuity may happen. For instance, consider a system that has to be turned off/on periodically, and that the turn-on time increases the chance of a failure in some components. Furthermore, the method presented here is a generalization, and properly works for absolute continuous cases too.

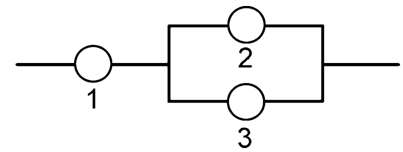

Fig. 4. System.

VI. Concluding Remarks, and Areas for Further Research

The novelty of this paper is the Bayesian nonparametric statistical analysis for parallel systems. Moreover, the generalization of the RHR using the product integral has allowed the development of the method. We have presented estimators for all reliability functions involved in parallel systems: distributions, sub-distributions, and RHR. The main result is the relationships among these important functions. We hope that these ideas can have strong impact on future research in engineering reliability.

To have coherent estimators, one does not need to be restricted to the case of absolutely continuous probability measures. In fact, we consider conditional continuous distributions given intervals between discontinuity points. We also rely on the fact that component lives are s-independent, and have disjoint sets of discontinuity points.

RHR, distribution, and sub-distribution functions in parallel systems play the same role of HR, survival, and sub-survival functions in series systems. Consequently, we have followed similar steps of the series system nonparametric Bayesian estimators.

The results presented here, together with those presented by Salinas-Torres *et al.* [7], allow one to think that any coherent system can have a nonparametric Bayesian estimation for all types of reliabilities. From Barlow & Proschan [8], we know that any coherent system is a combination of series, and parallel systems. For instance, it should be interesting to obtain similar results for the three-component system presented in Fig. 4.

Appendix I
Proof of Theorems

Proof of Theorem 1: Without loss of generality, we consider $j = 1$. Let t, h, and h^* be arbitrary positive numbers such that $h < h^*$. From the definition of $F_j^*(t)$, the difference

$$F_1^*(t+h) - F_1^*(t) = \Pr\left\{(t < X_1 \leq t+h) \bigcap_{\ell>1} (X_\ell < X_1)\right\} \tag{32}$$

has lower bound

$$\begin{aligned}&\Pr\{(t < X_1 \leq t+h) \cap_{\ell>1} (X_\ell < t)\}\\ &\quad = F_{1,\ldots,k}(t+h, t, \ldots, t) - F_{1,\ldots,k}(t, t, \ldots, t). \end{aligned} \tag{33}$$

Similarly, the upper bound of (32) is

$$\begin{aligned}\Pr\{(t < X_1 \leq t+h) \cap_{\ell>1} (X_\ell < t+h^*)\} \\ = F_{1,\ldots,k}(t+h, t+h^*, \ldots, t+h^*) \\ - F_{1,\ldots,k}(t, t+h^*, \ldots, t+h^*).\end{aligned} \tag{34}$$

Dividing (32)–(34) by h, and applying the limit with $h \to 0$, we obtain, for all $h^* > 0$,

$$\left.\frac{\partial F_{1,\ldots,k}(t_1, \ldots, t_k)}{\partial t_1}\right|_{t_1=t_2=\ldots=t_k=t} \leq \frac{\mathrm{d}F_1^*(t)}{\mathrm{d}t},$$

and

$$\frac{\mathrm{d}F_1^*(t)}{\mathrm{d}t} \leq \left.\frac{\partial F_{1,\ldots,k}(t_1, \ldots, t_k)}{\partial t_1}\right|_{t_1=t, t_2=\ldots=t_k=t+h^*}.$$

Again, taking the limit with $h^* \to 0$, we conclude the *Proof*:

$$\frac{\mathrm{d}F_1^*(t)}{\mathrm{d}t} = \left.\frac{\partial F_{1,\ldots,k}(t_1, \ldots, t_k)}{\partial t_1}\right|_{t_1=t_2=\ldots=t_k=t}. \tag{35}$$

■

Proof of Theorem 2: Taking $\sum$ over the set of all jump points of F_1^*, from (3), and (5), it is enough to show that

$$\oint_t^\infty \frac{\mathrm{d}F_1^*(y)}{F_1^*(y) + F_2^*(y)} = \oint_t^\infty \mu_1(y)\mathrm{d}y, \tag{36}$$

and

$$\sum_{y>t} \ln\left(\frac{F_1^*(y^-) + F_2^*(y^-)}{F_1^*(y^+) + F_2^*(y^+)}\right) = \sum_{y>t} \ln\left[1 - \mu_1(y)\right]. \tag{37}$$

Equation (36) is a consequence of properties 1, and 3,

$$\begin{aligned}\oint_t^\infty \frac{\mathrm{d}F_1^*(y)}{F_1^*(y) + F_2^*(y)} &= \oint_t^\infty \frac{F_2(y)\mathrm{d}F_1(y)}{F_1(y)F_2(y)} \\ &= \oint_t^\infty \frac{\mathrm{d}F_1(y)}{F_1(y)} \\ &= \oint_t^\infty \mu_1(y)\mathrm{d}y.\end{aligned}$$

Equation (37) is a consequence of

$$\begin{aligned}\sum_{y>t} \ln\left(\frac{F_1^*(y-) + F_2^*(y-)}{F_1^*(y+) + F_2^*(y+)}\right) &= \sum_{y>t} \ln\left(\frac{F_1(y-)F_2(y-)}{F_1(y+)F_2(y+)}\right) \\ &= \sum_{y>t} \ln\left(\frac{F_1(y-)}{F_1(y+)}\right) \\ &= \sum_{y>t} \ln\left[1 - \mu_1(y)\right].\end{aligned}$$

In the last equality, we use the fact that $F_2(y-) = F_2(y+)$ when y is a jump point of $F_1(\cdot)$. Because $F_1(\cdot)$ is positive and increasing, $F_1(t^*) = 0$ implies $F_1(t) = 0$ for $t < t^*$. ■

Proof of Theorem 3: To prove this theorem, replace appropriately (F_1, F_1^*, F_2, F_2^*) by $(F_\Delta, F_\Delta^*, F_{\Delta^c}, F_{\Delta^c}^*)$ in Theorem 2. ■

TABLE I
SIMULATED SAMPLE OF EXAMPLE 1

T	δ	T	δ	T	δ	T	δ	T	δ	T	δ
3.26	2	2.61	2	1.29	1	3.72	2	3.26	2	5.84	2
2.96	2	3.55	3	4.08	2	3.47	2	3.62	3	5.86	2
5.50	3	3.21	2	2.99	2	4.98	2	3.29	2	2.17	2
1.28	2	2.43	2	5.50	3	3.88	2	2.20	2	2.31	2
1.55	2	2.50	2	5.50	3	5.50	3	1.43	3	4.26	2
5.00	3	4.39	2	1.38	2	1.97	2	1.92	2	4.06	2
2.55	1	8.45	2	3.03	2	1.30	2	1.98	2	1.72	2
2.71	2	1.76	2	3.09	2	3.21	2	2.10	1	2.00	3
2.43	2	3.05	2	1.70	2	3.57	2	2.83	2	4.07	2
2.63	2	4.52	3	3.34	1	2.23	2	4.49	2	4.13	2
6.01	2	3.59	2	3.38	2	3.97	1	3.64	2	3.34	2
1.90	2	4.77	2	3.10	2	2.74	2	2.23	2	2.72	2
4.27	3	5.07	2	2.05	2	1.65	1	3.27	2	3.04	2
1.93	2	4.46	2	3.65	2	3.98	2	9.08	2	2.71	2
2.79	2	1.68	2	6.35	2	5.50	3	3.16	2	4.31	2
3.68	1	1.61	2	1.87	3	3.06	2	2.51	2	2.92	2
3.66	2	1.96	2	4.16	3	3.37	1				

TABLE II
SIMULATED SAMPLE OF EXAMPLE 2

T	δ	T	δ	T	δ	T	δ	T	δ	T	δ
3.72	2	5.92	2	4.36	3	2.18	3	3.96	2	5.00	3
4.59	2	5.81	2	4.88	3	4.42	3	5.00	3	2.25	3
3.82	2	3.23	3	3.34	3	2.74	3	4.09	3	4.04	3
4.52	3	3.29	3	3.61	3	4.90	3	4.45	2	4.59	3
2.99	2	4.76	1	2.99	2	3.17	1	3.27	3	4.27	3
3.36	2	4.99	2	4.18	1	3.03	3	5.12	2	3.27	3
5.00	3	3.55	3	2.81	2	3.73	2	4.84	3	3.83	3
4.57	3	5.00	3	3.71	2	2.99	3	5.00	3	4.49	2
2.57	2	4.73	3	4.26	3	3.23	3	5.00	3	2.72	3
3.31	2	3.43	2	1.86	2	3.13	2	4.89	3	3.43	3
3.33	2	2.80	2	4.21	2	4.22	3	5.03	2	3.47	2
5.00	3	4.78	2	4.16	3	4.84	3	2.72	2	4.29	3
3.08	2	3.66	3	3.17	3	3.43	3	3.18	2	3.49	3
3.48	3	2.59	3	5.74	2	4.88	3	4.25	2	2.81	3
4.07	3	5.00	3	4.05	2	1.77	2	3.39	3	1.79	2
2.48	2	2.66	2	4.41	2	5.00	3	3.81	3	3.33	2
3.55	2	3.93	2	4.39	2	7.38	2				

Proof of Theorem 4: Replacing the Bayesian estimates of F_Δ^*, and $F_{\Delta^c}^*$ in (21), we have

$$\widehat{F}_\Delta(t) = \exp\left\{\int_t^\infty \frac{-\mathrm{d}\widehat{F}_\Delta^*(s)}{\widehat{F}(s)}\right\} \prod_{s>t} \frac{\sum_{j=1}^k \widehat{F}_j^*(s-)}{\sum_{j=1}^k \widehat{F}_j^*(s+)}, \quad t \geq T_{(1)}, \tag{38}$$

where $\prod_{s>t}$ is the product over all jump points s of $\widehat{F}_\Delta^*$ with $s > t$.

Note that $\mathrm{d}\widehat{F}_\Delta^*(s) = \sum_{j\in\Delta} \mathrm{d}\alpha_j(0, s]/(c_k + n)$, and the first term in (38) becomes $I_\Delta(t)$. For each fixed $t > 0$, $\alpha_j(0, \cdot)$, $j = 1, \ldots, k$, are monotonic, continuous functions in (t, ∞), and $(1/\widehat{F}(\cdot))$ is monotonic in (t, ∞). Therefore, $\alpha_j(0, \cdot)$, $j = 1, \ldots, k$, can be decomposed uniquely as the difference of monotonic continuous functions (cf. Rudin [22], Corollary 1 of Theorem 6.27), and $(1/\widehat{F}(\cdot))$ as the difference of monotonic functions. Thus, the integral $\int_t^\infty (\sum_{j\in\Delta} \mathrm{d}\alpha_j(0, s]/\widehat{F}(s))$ is well defined. Moreover, the second factor in (38) is

$$\prod_{s>t} \frac{\sum_{j=1}^k \alpha_j(0, s] + \sum_{i=1}^n \mathbb{I}(T_i \leq s-)}{\sum_{j=1}^k \alpha_j(0, s] + \sum_{i=1}^n \mathbb{I}(T_i \leq s+)} = \Pi(t).$$

On the other hand, proceeding as in Lemma 1,

$$\widehat{F}_{\Delta}(t) = \mathrm{E}[F_{\Delta}(t)|\, data] = \underset{t}{\overset{\infty}{\pi}} \left(1 - \mathrm{d}\widehat{\mathcal{M}}_{\Delta}(s)\right), \quad (39)$$

where $\mathrm{d}\widehat{\mathcal{M}}_{\Delta}(s) = (c_k \mathrm{d}F^{*}_{\Delta,0}(s) + n\mathrm{d}\widehat{F}^{*}_{\Delta}(s)/c_k F_0(s) + n\widehat{F}(s))$. With simple algebraic manipulations, we obtain (38) from (39). ■

Appendix II
Simulated Samples

The simulated samples of $(T,\delta)_i$, $i = 1,\ldots,100$, used in the examples, are described in Tables I and II.

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Adriano Polpo received his BS degree in Statistics at the State University of Campinas (Brazil) in 2001, and the PhD degree in Statistics at the University of São Paulo (Brazil) in 2005. Dr. Polpo is presently an Assistant Professor at the Federal University of São Carlos (Brazil). His current interests are reliability, and Bayesian and nonparametric methods.

Carlos A. B. Pereira received his BS degree in Statistics at the National School of Statistical Science (Brazil) in 1968, MS degree in Statistics at the University of São Paulo (Brazil) in 1971, MS degree in Statistics at the Florida State University (USA) in 1978, and the PhD degree in Statistics at the Florida State University (USA) in 1980. Dr. Pereira is presently a Full Professor of Statistics, and Bioinformatic Nucleus Director at the University of São Paulo. His current interests are Bayesian prediction, hypothesis tests, and Bioinformatic. Dr. Pereira has published seven books, and more than 100 papers.